it need not occupy its limiting position LL'', and its motion is therefore discontinuous. The following examples will illustrate this:

(a) Let P and P' move on two straight lines c and c' (Fig. 2), which intersect at L, in such a way that the ratio $LP : LP'$ is constant; and let P'' move on a line c'' which does not pass through L. The line l moves parallel to itself; and as P and P' approach coincidence at L, l approaches the limit LL''. But as l is not determined at L, its motion is not continuous there.

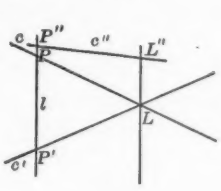


FIG. 2.

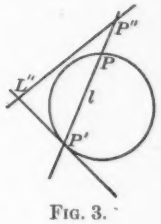


FIG. 3.

(b) P moves on a circle c, and P' is a fixed point on c (Fig. 3). As P moves into coincidence with P', l approaches the limiting position $P'L''$ which is defined to be the tangent to the circle at P'. But l is not determined when P is at P', and the rotation of l about P' at this point is therefore discontinuous.

If we fix, by definition, the position of l when P coincides with P' to be that of the tangent at P', we make the rotation of l about P' to be continuous at every point as P moves over the whole circumference.

4. Variable quantities.

The movement of points and lines in the plane implies the variation of quantities such as length, area, and angle. Thus if A of the triangle ABC moves in the plane while B and C remain fixed, the lines AB and AC rotate about B and C; and the lengths AB and AC, the area ABC, and the angles A, B, and C are variable quantities.

The definitions of § 2 practically involve the following definitions concerning variable quantities:

(i) If the value of a variable quantity, as it approaches a fixed value v, may be made to differ from v by less than any assignable quantity however small, whether it approaches through values greater or less than v, then v is the *limit* of the values of the quantity as it approaches v. (Cf. Goursat-Hedrick, *Mathematical Analysis*, vol. I, § 1.)

(ii) If the limit of a variable quantity and its value at a point are the same, the variation is continuous at that point. (Cf. Pierpont, *l. c.*, vol. I, § 339.)

(iii) The variation of a quantity is continuous over any interval when it is continuous at every point of that interval. (Cf. Goursat-Hedrick, *l. c.*, vol. I, § 3.)

Thus for example in Fig. 1 the length AP is continuous over the interval from 0 to AB in 2 (iii) (a); but it is discontinuous at D in (b), and at E and D in (c), as P moves from A to B.

5. In general, continuous movements of points and lines in the plane imply continuity in the variation of the quantities involved. Thus, for example, the distance AP from a fixed point A to a point P which moves continuously along a

curve is a continuous variable. Similarly the angle AOP between a line OP, which rotates continuously about a fixed point O, and a fixed initial position OA is a continuous variable. In each case the variable attains its limiting value when the point or the line reaches its limiting position.

But examples are given in 3 (iv) (a) and 3 (iv) (b) of continuous movements of points which imply discontinuous movements of lines and hence discontinuous variation in angle. In these cases the points reach their limiting positions where at the same time the angle need not attain its limiting value.

Similarly a continuous rotation of a line may determine a discontinuous movement of a point. Thus let O be a fixed point and l a fixed line (Fig. 4); and let a line m rotate continuously about O and intersect l in P. As m rotates about O, P moves continuously along l except when l and m are parallel. The movement of P and the distance AP are discontinuous at this point.

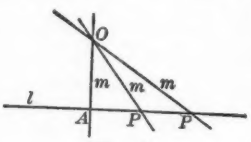


FIG. 4.

On the other hand, if the point at infinity be defined as usual as a unique point on l, the movement of P along l, which is discontinuous when P is the point at infinity, implies a continuous rotation of m about O.

6. In a plane geometric figure some points may be considered fixed while others vary, as for example in the triangle ABC of § 4. Similarly in the quadrangle $ABCD$ we may assume the points A, B, C to remain fixed and D to move freely in the plane. The form of the quadrangle then changes. The angles at C, D, A and the lengths CD and DA are variable quantities, and are functions of D in the sense that their values depend on the position of D. If D moves continuously these quantities vary continuously, except possibly where D coincides with one of the other points. When D approaches a limit, all the variable elements which depend on it approach at the same time their respective limits; and if no discontinuities exist they all have the values of their limits when D occupies its limit.

As D moves along the curve DD'' in Fig. 5, the angle BCD changes to BCD''. In its variation it passes through a zero value when D is at D'; and on the principle that a continuously increasing or decreasing variable changes sign on passing through a zero value, we say that BCD and BCD'' have opposite signs.

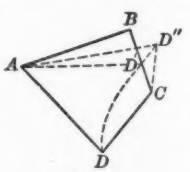


FIG. 5.

7. Suppose a finite set of variables to approach their respective limits at the same time; and suppose a constant relation to exist among these variables for all simultaneous sets of values preceding the limit. Then this relation will exist also at the limit, provided no discontinuities appear. This is evident, for it is merely stated that a certain relation exists among a set of values when it exists for values which differ from them by as little as we please. The principle thus stated is important in that it leads to generalizations among theorems of geometry. The following examples are given in illustration:

(i) ABC is a triangle inscribed in a circle, with A and B fixed, and C variable, on the circumference (Fig. 6). Then, from elementary geometry, the angle ACB is constant for all positions of C provided C remains on either the one side or on the other of AB.

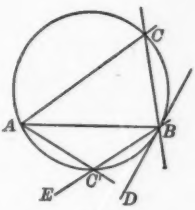


FIG. 6.

But assuming, as in 3 (iv) (*b*), that when C is at B, CB is the tangent BD at B, and that therefore all the elements involved are continuous as C moved into coincidence with B, we have from the above principle that the angle ABD is equal to the constant angle C. Similarly the angle $AC'E$ is seen to be equal to ABD. It follows therefore that the angle ACB is constant as C moves about the whole circumference.

This general statement includes the several theorems usually given in this connection in elementary geometry. It depends however on the assumption that CB occupies its limiting position BD when C comes to B.

The theorems thus connected are:

(*a*) Two angles at the circumference of a circle, standing on the same arc, are equal.

(*b*) The angles between a tangent and a chord from the point of contact are equal to the angles in the alternate segments.

(*c*) An exterior angle of a quadrilateral inscribed in a circle is equal to the opposite interior angle.

(ii) The rectangles on the segments of two intersecting chords of a circle are equal; that is, $PA \cdot PB = PC \cdot PD$.

By moving P continuously, we find that the theorem takes the following additional forms:

(*a*) The square on the ordinate from a point on the diameter of a circle is equal to the rectangle on the segments of the diameter.

(*b*) The rectangles on the segments of two intersecting secants are equal.

(*c*) If a secant and a tangent be drawn from the same point, the rectangle on the segments of the secant is equal to the square on the tangent.

(*d*) The squares on the two tangents from a point are equal.

(iii) The square on a side of a triangle, opposite an acute angle, is equal to the sum of the squares on the other two sides diminished by twice the rectangle on one of these and the projection of the other on it; that is, $AB^2 = BC^2 + CA^2 - 2BC \cdot DC$.

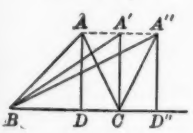


FIG. 7.

The principle of this theorem includes also each of the following:

(*a*) When A moves to D (Fig. 7), the line segment BA is divided externally at C; and for this divided line segment, $BA^2 = BC^2 + CA^2 - 2BC \cdot CA$.

(*b*) When A moves to A', and DC vanishes, the theorem is: The square on the hypotenuse of a right angled triangle is equal to the sum of the squares on the other two sides.

(*c*) When A moves to A'', $D''C$ is negative and the angle BCA'' is obtuse. Then $A''B^2 = BC^2 + CA''^2 + 2BC \cdot CD''$.

(*d*) When A moves to D'', the line segment BA is divided internally at C; and for this divided segment $BA^2 = BC^2 + CA^2 + 2BC \cdot CA$.

(iv) $ABCD$ is a complete quadrangle with opposite pairs of sides AB and CD, AC and BD, and AD and BC; and E and F are the middle points of one of these pairs, say BD and AC respectively. Then from elementary geometry,

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4EF^2.$$

This theorem, in principle, includes the following:

(*a*) When D moves to B, $AB^2 + BC^2 = 2BF^2 + 2AF^2$. That is, the sum of the squares on two sides of a triangle is equal to twice the square on half the third side and twice the square on the median to the third side.

(*b*) When B and D move to H, any point on the llne-segment AC, $AH^2 + HC^2 = 2AF^2 + 2FH^2$. That is, if a line segment be divided equally at one point and unequally at another, the sum of the squares on the unequal segments is equal to twice the sum of the squares on half the line and on the part between the points of section.

(*c*) When D moves to C, and E therefore to the middle point G of BC, $AB^2 = 4FG^2$. That is, the square on the base of a triangle is equal to four times the square on the line joining the middle points of the sides.

Various other properties of the triangle and of the divided line segment may be obtained as particular cases of this general theorem.

In the same way also properties of any polygon may be found in special forms as properties of any polygon of a smaller number of sides, or of the divided line segment. Examples (iii) and (vii) further illustrate the same idea.

(v) The three radical axes of three circles taken two and two are concurrent.

This theorem takes particular forms as one, two, or three of the circles become points.

If the three circles become points, the theorem states that the right bisectors of the sides of a triangle are concurrent.

(vi) S is a circle, tangent to four mutually external circles S_1, S_2, S_3, and S_4; and t_{12} is a common tangent to S_1 and S_2, etc. Then by Casey's theorem[1]

$$t_{12}t_{34} \pm t_{13}t_{42} \pm t_{14}t_{23} = 0.^2$$

This theorem takes particular forms as one, two, three, or four of the circles become points. Thus, for example,

(*a*) If the four circles which touch S become points, the result is Ptolemy's theorem on the concyclic quadrangle.

[1] *Proceedings of the Royal Irish Academy*, 1866. See also J. L. Coolidge, *A Treatise on the Circle and Sphere*, Oxford, 1916, p. 38.—EDITOR.

[2] "Here all the t_{ij}'s denote common direct tangential segments, or those connecting two pairs [of circles] with no common member denote direct tangents and the other four transverse, or those which lack one subscript denote direct, and those which include it transverse tangential segments" (Coolidge).

(*b*) If the four circles which touch S become points and S becomes a straight line, we have the common theorem on four collinear points A, B, C, D, viz: $AB \cdot CD + BC \cdot AD + CA \cdot BD = 0$.

(vii) A, B, C, D, E, F, are six points in any order on a conic section, and $ABCDEFA$ is an inscribed hexagram. If AB and DE, BC and EF, CD and FA intersect in X, Y, Z, respectively, then XYZ is a straight line, the Pascal line of the hexagram.

By moving vertices into coincidence, and assuming that their joins become tangents to the conic, we obtain this theorem in forms adapted to the pentagram, the tetragram, and the triangle.

Thus, for example, the hexagram becomes a triangle if B moves to A, D to C, and F to E. The theorem then becomes: If a triangle be inscribed in a conic, the sides intersect the tangents at the opposite vertices collinearly. The line of collinearity is the Pascal line of the triangle. Or the theorem may be stated thus: If a triangle be inscribed in a conic, and another circumscribed at the vertices of the first, the two triangles are in perspective. The axis of perspective is the Pascal line of the inscribed triangle.

This list of examples may be increased indefinitely from both elementary and advanced geometry. It will be found everywhere that general principles may be discovered to underlie groups of theorems which in our ordinary teaching of the subject have had no relationship to one another.

A NEW PROOF OF THE LAW OF TANGENTS.[1]

By WM. F. CHENEY, JR., Berkeley, Cal.

1. In the triangle ABC, with angles A, B, and C, and sides a, b, and c, assume b greater than c. Lay off AD along b, equal to c, and draw BD; then

$$\angle DBA = \tfrac{1}{2}(180^\circ - A) = \tfrac{1}{2}(B + C);$$

[1] Other geometrical discussions of the law of tangents may be found in the following sources: W. E. Johnson, *Treatise on Trigonometry*, London, 1889, p. 96; R. Levett and C. Davison, *Elements of Plane Trigonometry*, London, 1892, pp. 170–171 (also in J. W. Mercer, *Trigonometry for Beginners*, Cambridge, 1906, pp. 259–260); E. Brand, *Journal de mathématiques élémentaires* (de Longchamps), 1895, pp. 153–154; E. M. Langley, *Journal de mathématiques élémentaires* (de Longchamps), 1896, pp. 3–4 (construction introducing Wallace's Line); E. W. Hobson, *A Treatise on Plane Trigonometry*, second edition, Cambridge, 1897, pp. 155–156; E. J. Wilczynski, *Plane Trigonometry and Applications*, Boston, 1914, pp. 105–106; and J. W. Young and F. M. Morgan, *Plane Trigonometry and Numerical Computation*, New York, 1919, pp. 47–48.

The law of tangents in the usual form (except for notation) was first given by Vieta in the seventeenth century; *Francisci Vietæ opera mathematica*, Lugduni Batavorum, 1646, p. 402: "Vt adgregatum crurum ad differentiam eorundem, ita prosinus dimidiæ summæ angulorum ad basin ad prosinum dimidiæ differentiæ."

Cf. A. von Braunmühl, *Vorlesungen über Geschichte der Trigonometrie*, Teil 1, 1900, p. 188; Teil 2, pp. 44–45; and other places referred to under heading "Tangentensatz" in indexes.—EDITOR.

and

$$\angle CBD = \tfrac{1}{2}(B - C).$$

2. Draw AP, the perpendicular bisector of BD, and produce it to meet BC at Q; then

$$AP = BP \cdot \tan \tfrac{1}{2}(B + C);$$

and

$$PQ = BP \cdot \tan \tfrac{1}{2}(B - C).$$

3. Draw PR parallel to BC, bisecting CD at R; then

$$AR = \tfrac{1}{2}(b + c);$$

and

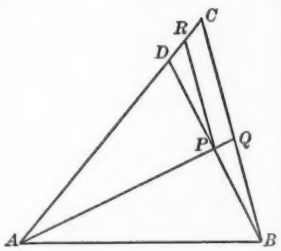
$$RC = \tfrac{1}{2}(b - c).$$

4. But

$$\frac{AR}{RC} = \frac{AP}{PQ};$$

therefore

$$\frac{b + c}{b - c} = \frac{\tan \tfrac{1}{2}(B + C)}{\tan \tfrac{1}{2}(B - C)}.$$



QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

DISCUSSIONS.

Professor J. E. Trevor gave an example in the December number of the MONTHLY of an instance in which a theorem of analysis is of importance in thermodynamics. Another case of this sort is contributed by him as the first discussion in the present number. Corresponding to a change of independent variable, the form of a function is altered; it is readily shown that the rate of change can be represented as an ordinary derivative of another function closely related to the first. The notion is of use in connection with the heats of dilution of a solution. Instances in which mathematical conceptions arise in connection with applied science are always of value; even those mathematicians whose personal interests are principally in so-called "abstract" fields cannot afford to neglect the phases of mathematics sometimes termed "practical."

In the study of analytic geometry beyond the more elementary portions determinants play an extremely important part; a glance at any treatise on projective or metric analytic geometry, on differential geometry, or on non-Euclidean geometry analytically treated, will reveal page after page filled with resultants, discriminants, Hessians, linear dependence, and other ideas based on determinants. In the elementary parts of the subject, however, determinants

play a negligible rôle. In most texts in common use, the determinantal form for the area of a triangle is a solitary instance. Professor Foraker in the second discussion below gives an introduction to the use of determinants in the elementary field. Most of his work consists of the statement of formulae; one theorem is proved, relating to the collinearity of the circumcenter, centroid, and orthocenter of a triangle. It is scarcely to be expected that the material would be suitable for ordinary class use, but it will be of interest to a few students in any class.

The third discussion is a plea for the more general use of the history of mathematics in connection with secondary and collegiate instruction. Attracted by an announcement of the plans of the National Committee on Mathematical Requirements, Mr. Laurin Zilliacus of the Bedales School in Petersfield, England, has written a letter on the subject to Professor H. W. Tyler, a member of the committee. With the permission of the writer, the recipient, and Professor J. W. Young, chairman of the committee, we reproduce nearly the entire letter, retaining the personal form in which it is written. The fact that Mr. Zilliacus has actual experience of the growth of interest and enthusiasm in a class through the use of historical material is of more value than any amount of mere theorizing on the question.

Another article basing pedagogic recommendations on the result of personal experience is found in the last discussion, by Professor Ettlinger, on the use of graphical methods in trigonometry. It is nearly certain that all teachers make some use, in their teaching, of the ideas suggested by Professor Ettlinger; not all, however, have carried these ideas so far, or treated them with such emphasis. One incidental point in the paper moves the editor to a comment which he has long hoped to see made. It is pointed out that a correctly drawn figure is of great aid in attacking a problem. This is undeniably true in trigonometry, and especially in elementary geometry. Is it not also true, on the other hand, that sometimes a figure drawn with carefully planned *inaccuracy* is of extreme importance and inspiration?

I. Heats of Dilution.

By J. E. Trevor, Cornell University.

The "heats of dilution" of a solution are mathematical curiosities, in that these quantities are defined with reference to changes of the form of a function. When $\psi(x)$ is a given function, and δx is an arbitrary positive increment of the independent variable x, let it be supposed that a certain operation changes the value of a quantity φ from the value

$$\varphi_1 = \psi(x) + a \cdot \delta x$$

to the value

$$\varphi_2 = \psi(x + \delta x),$$

where a is a constant. The change of value $\varphi_2 - \varphi_1$ is due to the change of

form of the function $\varphi(x, \delta x)$, and the rate of change of φ per unit increment of x in $\psi(x)$ is

$$\lim_{\delta x \to 0} \frac{\psi(x + \delta x) - \psi(x) - a \cdot \delta x}{\delta x} = \frac{d(\psi - ax)}{dx}.$$

The heats of dilution of a solution are defined by such limits.

Consider constant masses M_1, M_2 of two component substances, such as salt and water or water and alcohol, capable of forming a homogeneous liquid mixture. Let the state of the body constituted of the masses M_1, M_2 be such that any mass m_j of the jth component exists separately from the mixture of the masses $M_j - m_j$ and M_k, where M_k is the mass of the other component and $M_j \geq m_j \geq 0$. In the case $m_j = 0$, *i.e.*, when the body is a homogeneous "solution" σ in a state of stable thermodynamic equilibrium under the pressure p at the temperature θ, let $E(p, \theta, M_1, M_2)$ be the energy and $V(p, \theta, M_1, M_2)$ be the volume of the solution, where p, θ, M_1, M_2 are variable parameters. When the body is in a state σ_1 with separated parts, each in stable equilibrium at p, θ, its energy E_1 and volume V_1 are the sums of the energies and volumes of the parts. The "enthalpy" G of the solution σ and the enthalpy G_1 of the body in the state σ_1 are defined by the equations

$$G = E + pV, \qquad G_1 = E_1 + pV_1,$$

from which it follows that

$$E - E_1 = (G - G_1) - p(V - V_1).$$

When the separate parts are brought together, under the constant pressure p, the state σ_1 is transformed into the state σ and the work $-p(V - V_1)$ is absorbed by the body. Hence, by the energy law, the heat absorbed by the body is the quantity $G - G_1$. The particular case of this process realized when $m_j = M_j$ is the formation of the solution from its components. If g_j is the enthalpy of unit mass of the jth component, the enthalpy G_1 of the separate components is $M_1g_1 + M_2g_2$, and the "heat of mixing" ΔG of the solution is

$$\Delta G = G - M_1g_1 - M_2g_2, \tag{1}$$

where g_1, g_2 are functions of p, θ.

Consider now the dilution of the solution whose mass is $M_1 + M_2$, by addition of the mass δM_j of the jth component at p, θ. The value of the enthalpy of the body composed of the masses $M_1 + M_2$ and δM_j is

$$G(p, \theta, M_1, M_2) + g_j(p, \theta)\delta M_j \tag{2}$$

before the operation, and it is

$$G(p, \theta, M_j + \delta M_j, M_k) \tag{3}$$

after the operation. Both before and after the operation, the enthalpy of the body is a function of the constant quantities $p, \theta, M_j, M_k, \delta M_j$. In the opera-

tion the enthalpy of the body changes in value because the form of the function changes from the form (2) to the form (3). Suppressing the constants p, θ, M_k, this change of value is

$$G(M_j + \delta M_j) - G(M_j) - g_j \delta M_j.$$

Since this expression denotes the heat absorbed by the body $(M_j, M_k, \delta M_j)$ during the dilution, we have that the heat absorbed per unit mass of diluent added is the limit

$$\lim_{\delta M_j \to 0} \frac{G(M_j + \delta M_j) - G(M_j) - g_j \delta M_j}{\delta M_j} = \frac{\partial G}{\partial M_j} - g_j.$$

This quantity is the "heat of dilution" Δ_j of the solution (M_1, M_2) for dilution by the jth component of the mixture.

Now by differentiating (1) we find

$$\frac{\partial \Delta G}{\partial M_j} = \frac{\partial G}{\partial M_j} - g_j.$$

Hence Δ_j is equal to the derivative $\partial \Delta G / \partial M_j$ of the heat of mixing. Further, since it is known that $G(p, \theta, M_1, M_2)$ is homogeneous of degree one in M_1, M_2, and hence by (1) that ΔG is homogeneous of degree one in these variables, we have

$$\Delta G = M_1 \Delta_1 + M_2 \Delta_2.$$

It thus appears that the heat of mixing of a solution is a linear function of the two heats of dilution.

II. Determinants in Elementary Analytic Geometry.

By F. A. Foraker, University of Pittsburgh.

The general purpose of this paper is to indicate how some results in elementary analytic geometry may be conveniently expressed in determinant form. We shall consistently use the notation

$$(1) \qquad \Delta = \begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix}, \qquad D = \begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix};$$

and shall in the usual way indicate the cofactors of elements of determinants by corresponding capital letters. The points P_1, P_2, P_3 will be understood to have coördinates (x_1, y_1), (x_2, y_2), (x_3, y_3) respectively. It is well known that the area of the triangle $P_1P_2P_3$ is $\frac{1}{2}\Delta$, and that therefore the points P_1, P_2, P_3 are collinear if and only if $\Delta = 0$.

The equation of the line joining P_1 and P_2 is

$$(2) \qquad \begin{vmatrix} x, & y, & 1 \\ x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \end{vmatrix} = 0;$$

in particular, the equation of a line having intercepts a, b is

$$\begin{vmatrix} x, & y, & 1 \\ a, & 0, & 1 \\ 0, & b, & 1 \end{vmatrix} = 0. \tag{3}$$

The equations of lines respectively parallel and perpendicular to (2) through P_3 are

$$\begin{vmatrix} x - x_3, & y - y_3, & 0 \\ x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \end{vmatrix} = 0 \tag{4}$$

and

$$\begin{vmatrix} y_3 - y, & x - x_3, & 0 \\ x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \end{vmatrix} = 0. \tag{5}$$

The latter equation may also be written:

$$\begin{vmatrix} x, & y, & 1 \\ x_3, & y_3, & 1 \\ X_3, & Y_3, & 0 \end{vmatrix} = 0. \tag{6}$$

The equation of a line through P_1 with slope m is

$$\begin{vmatrix} x, & y, & 1 \\ x_1, & y_1, & 1 \\ 1, & m, & 0 \end{vmatrix} = 0; \tag{7}$$

the normal form of the equation of a line may be written

$$\begin{vmatrix} x, & y, & 1 \\ p, & 0, & \cos\alpha \\ 0, & p, & \sin\alpha \end{vmatrix} = 0, \tag{8}$$

provided $p \neq 0$.

All these equations are of the general type

$$\begin{vmatrix} x, & y, & 1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix} = 0, \tag{9}$$

or may be readily reduced to that type. Conversely, (9) represents a straight line unless $A_1 = B_1 = 0$. From (9) any of the usual forms may be derived; it is seen at once that the intercepts and the slope are

$$a = -C_1/A_1, \qquad b = -C_1/B_1; \qquad m = -A_1/B_1.$$

The equation of a circle through the points P_1, P_2, P_3 is

$$\begin{vmatrix} x^2 + y^2, & x, & y, & 1 \\ x_1^2 + y_1^2, & x_1, & y_1, & 1 \\ x_2^2 + y_2^2, & x_2, & y_2, & 1 \\ x_3^2 + y_3^2, & x_3, & y_3, & 1 \end{vmatrix} = 0; \tag{10}$$

this may be written

$$\Delta(x^2+y^2) - |\,x_i^2+y_i^2,\ y_i,\ 1\,|\,x - |\,x_i,\ x_i^2+y_i^2,\ 1\,|\,y = |\,x_i^2+y_i^2,\ x_i,\ y_i\,|\,,$$

where the meaning of the abbreviated determinants is obvious. Thus the circumcenter of the triangle $P_1P_2P_3$ has the coördinates

$$(11)\qquad C_x = \frac{1}{2\Delta}|\,x_i^2+y_i^2,\ \ y_i,\ \ 1\,|\,,\qquad C_y = \frac{1}{2\Delta}|\,x_i,\ \ x_i^2+y_i^2,\ \ 1\,|\,.$$

The median through the vertex P_1 of the triangle $P_1P_2P_3$ has the equation

$$(12)\qquad \begin{vmatrix} x, & y, & 1 \\ x_1, & y_1, & 1 \\ x_2+x_3, & y_2+y_3, & 2 \end{vmatrix} = 0,$$

which may be written in the form

$$\begin{vmatrix} x, & y, & 1 \\ x_1, & y_1, & 1 \\ x_1+x_2+x_3, & y_1+y_2+y_3, & 3 \end{vmatrix} = 0.$$

From symmetry it is seen at once that the three medians intersect in a single point, the centroid, having the coördinates

$$(13)\qquad G_x = \tfrac{1}{3}(x_1+x_2+x_3),\qquad G_y = \tfrac{1}{3}(y_1+y_2+y_3).$$

The altitudes of the triangle $P_1P_2P_3$ through P_1 and P_2 have respectively the equations

$$\begin{vmatrix} x, & y, & 1 \\ x_1, & y_1, & 1 \\ y_2-y_3, & x_3-x_2, & 0 \end{vmatrix} = 0,\qquad \begin{vmatrix} x, & y, & 1 \\ x_2, & y_2, & 1 \\ y_1-y_3, & x_3-x_1, & 0 \end{vmatrix} = 0.$$

Solving, we find

$$x = \frac{\begin{vmatrix} x_1x_3-x_1x_2+y_1y_3-y_1y_2, & y_3-y_2 \\ x_2x_3-x_1x_2+y_2y_3-y_1y_2, & y_3-y_1 \end{vmatrix}}{\begin{vmatrix} x_3-x_2, & y_3-y_2 \\ x_3-x_1, & y_3-y_1 \end{vmatrix}}.$$

If we interchange rows, change the signs in certain columns, and border with the quantities indicated, we have

$$x = \frac{\begin{vmatrix} x_2x_3-x_1x_2+y_2y_3-y_1y_2, & y_1-y_3, & 0 \\ x_1x_3-x_1x_2+y_1y_3-y_1y_2, & y_2-y_3, & 0 \\ x_1x_2+y_1y_2, & y_3, & 1 \end{vmatrix}}{-\begin{vmatrix} x_1-x_3, & y_1-y_3, & 0 \\ x_2-x_3, & y_2-y_3, & 0 \\ x_3, & y_3, & 1 \end{vmatrix}};$$

so that finally

$$x = \frac{\begin{vmatrix} x_2x_3 + y_2y_3, & y_1, & 1 \\ x_3x_1 + y_3y_1, & y_2, & 1 \\ x_1x_2 + y_1y_2, & y_3, & 1 \end{vmatrix}}{-\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}}.$$

In like manner we may obtain y. The results involve the three subscripts symmetrically; therefore the point must be the intersection of each pair of altitudes. As the common intersection of the three altitudes, the orthocenter, we thus have

$$O_x = -\frac{1}{\Delta}\begin{vmatrix} x_2x_3 + y_2y_3, & y_1, & 1 \\ x_3x_1 + y_3y_1, & y_2, & 1 \\ x_1x_2 + y_1y_2, & y_3, & 1 \end{vmatrix},$$

(14)

$$O_y = -\frac{1}{\Delta}\begin{vmatrix} x_1, & x_2x_3 + y_2y_3, & 1 \\ x_2, & x_3x_1 + y_3y_1, & 1 \\ x_3, & x_1x_2 + y_1y_2, & 1 \end{vmatrix}.$$

From (11), (13), (14) we have

$$S = \Delta(3G_x - 2C_x - O_x) = \begin{vmatrix} r_1 & y_1 & 1 \\ r_2 & y_2 & 1 \\ r_3 & y_3 & 1 \end{vmatrix},$$

where

$$\begin{aligned} r_1 &= x_1(x_1 + x_2 + x_3) - (x_1^2 + y_1^2) + (x_2x_3 + y_2y_3) \\ &= (x_2x_3 + x_3x_1 + x_1x_2) + (y_2y_3 - y_1^2), \end{aligned}$$

with similar expressions for r_2, r_3. Hence

$$S = (x_2x_3 + x_3x_1 + x_1x_2)\begin{vmatrix} 1, & y_1, & 1 \\ 1, & y_2, & 1 \\ 1, & y_3, & 1 \end{vmatrix} + \begin{vmatrix} y_2y_3, & y_1, & 1 \\ y_3y_1, & y_2, & 1 \\ y_1y_2, & y_3, & 1 \end{vmatrix} - \begin{vmatrix} y_1^2, & y_1, & 1 \\ y_2^2, & y_2, & 1 \\ y_3^2, & y_3, & 1 \end{vmatrix}.$$

The first determinant vanishes; each of the others is equal to $(y_3 - y_2)(y_1 - y_3)(y_2 - y_1)$; therefore $S = 0$. We may similarly examine the y-coördinates; we find that

$$\begin{aligned} 3G_x - 2C_x - O_x &= 0, \\ 3G_y - 2C_y - O_y &= 0. \end{aligned}$$

Thus we have proved the theorem: *The centroid lies one-third of the distance from the circumcenter to the orthocenter, along the line joining them.*

If we have the equations of three lines given in the form

$$a_ix + b_iy + c_i = 0 \qquad [i = 1, 2, 3], \tag{15}$$

the lines will form a triangle if D, C_1, C_2, C_3 are all different from zero. The

vertices of the triangle are then

$$(16)\qquad \begin{aligned} x_{23} &= A_1/C_1, & y_{23} &= B_1/C_1.\\ x_{31} &= A_2/C_2, & y_{31} &= B_2/C_2.\\ x_{12} &= A_3/C_3, & y_{12} &= B_3/C_3. \end{aligned}$$

The coördinates of the circumcenter are seen by (11) and (16) to be[1]

$$(17)\qquad \begin{aligned} C_x &= \frac{1}{2D^2}\begin{vmatrix} \dfrac{A_i^2+B_i^2}{C_i}, & B_i, & C_i \end{vmatrix},\\ C_y &= \frac{1}{2D^2}\begin{vmatrix} A_i, & \dfrac{A_i^2+B_i^2}{C_i}, & C_i \end{vmatrix}. \end{aligned}$$

The coördinates of the centroid are

$$(18)\qquad C_x = \tfrac{1}{3}\left(\frac{A_1}{C_1}+\frac{A_2}{C_2}+\frac{A_3}{C_3}\right), \qquad C_y = \tfrac{1}{3}\left(\frac{B_1}{C_1}+\frac{B_2}{C_2}+\frac{B_3}{C_3}\right).$$

The coördinates of the orthocenter are

$$(19)\qquad \begin{aligned} O_x &= -\frac{1}{C_1C_2C_3D^2}\begin{vmatrix} (A_2A_3+B_2B_3)C_1^2, & B_1, & C_1\\ (A_3A_1+B_3B_1)C_2^2, & B_2, & C_2\\ (A_1A_2+B_1B_2)C_3^2, & B_3, & C_3 \end{vmatrix},\\ O_y &= -\frac{1}{C_1C_2C_3D^2}\begin{vmatrix} A_1, & (A_2A_3+B_2B_3)C_1^2, & C_1\\ A_2, & (A_3A_1+B_3B_1)C_2^2, & C_2\\ A_3, & (A_1A_2+B_1B_2)C_3^2, & C_3 \end{vmatrix}. \end{aligned}$$

The area of the triangle is $D^2/(2C_1C_2C_3)$.

A few of the above results are well known; the others are easily proved. This paper is intended only as an introduction to work along this line and is not supposed to be exhaustive in any sense.

III. The History of Mathematics in Elementary Instruction.[2]

By Laurin Zilliacus, Bedales School, Petersfield, England.

The General Education Board has recently announced the appropriation of $16,000 to be used by the National Committee on Mathematical Requirements in financing a study looking to improvements in the mathematical curriculum of the secondary schools of America. If there is to be any recasting of curricula, I earnestly hope that a full use will be made of the history of mathematics. A short perusal of this history, as given in popular works such as Ball's *History of Mathematics*, would, I feel sure, cause a majority of present-day teachers of mathematics to modify considerably both the outlines of their courses and their

[1] Since $|A_1B_2C_3| = |a_1b_2c_3|^2$.

[2] Extract from a letter to Professor H. W. Tyler.

methods of presenting the subject. The particular lines of change which would result I am not qualified to predict, but I think it would be safe to say in general that (1) there would be a reduction in unnecessary drudgery as a true perspective of values was obtained; (2) the fundamental principles (*e.g.*, compact number notation, symbolic statement, variation, limits, deductive reasoning) would stand out more clearly and comprehensively when it was seen how they had been gradually pulled out from confusion, instead of their being, as at present, presented as part of a complete and arbitrary structure; (3) there would be a living connection between mathematics and "practical" life in the minds of children who were shown how each has waited on the other throughout the ages.

My own experience in a school of boys and girls of 11 to 19 years of age has convinced me that the coördination of mathematics with experimental science and the historical treatment of mathematics go hand in hand and together yield most fruitful results. A couple of lectures in the school on the history of mathematics roused great interest and led to many questions and some independent reading. A chart was displayed of the "Stream of Mathematical Knowledge" from the earliest known times, showing how the thin trickle of isolated facts from the Egyptians was swelled by the Greeks into a great river of knowledge; how the river branched and wandered, increasing here and shrinking there, through the Arabs (with a great tributary from the Hindoos), the Romans, the Byzantines and the Moors, until with the advent of the printing press and the flight of the Byzantine Greeks from Constantinople the streams reunited in Europe, and swelled on every hand—there the chart ended, but it is to be continued later. I found that the children were delighted as with a story, and that mathematics acquired a new meaning for them when linked up with historical familiars such as Athens, Alexandria, the Mohammedan conquests, Haroun al Raschid, the Moors of Spain, etc., etc.

In particular classes, too, the history was of great service. My difficulties with a Mechanics class vanished when a discovery in books (Cox's *Mechanics*) enabled me to outline the development of our views of the solar system from the theory of epicycles through the Copernican system, Galileo's trouble-bringing discoveries and Kepler's Laws to Newton's great work. The class was deeply interested, and grasped Newton's Laws, circular motion, energy and momentum in a way I have never before experienced with a class.

A beginner's class in geometry was much helped by a short account of how Thales and his successors took experimentally determined facts from the Egyptians, investigated their explanation and deduced other useful facts from them: the meaning of a proof was quickly grasped, and very rapid progress made in what they regarded as an intellectual game. They themselves discovered experimentally the property of angles in a semicircle, and were probably as eager as Thales to find the explanation and proof.

Summing up the benefits that struck me even during my short experience of teaching mathematics with a due regard for its history, I should say they are as follows:

For the teacher: (1) The relative importance of various aspects of the subject becomes easier to determine, and hence what should be omitted and what emphasized.

(2) Indications are obtained of the order in which to introduce new knowledge, since the history of the world's progress in mathematics is not without similarity to that of an individual (*e.g.*, the late appearance in history of an understanding of the minus sign in algebra indicates where it should be attempted in the school study of this subject).

(3) The difficulties of beginners are more readily seen, for they are much the same throughout time. (A striking instance of this is the confusion between determinate and indeterminate problems—once pointed out, it never recurs.)

For the taught: (4) An understanding of mathematics as a body of methods and knowledge that have grown and are still growing by the labours of men, and not as a curiously pointless and arbitrary set of rules and definitions.

(5) A better grasp of underlying principles as the course of history makes them stand out more and more clearly.

(6) A connection of mathematics with interesting and useful inventions.

(7) The interest that any subject gains that is packed with anecdotes and stories.

IV. An Introduction to Plane Trigonometry by Graphical Methods.

By H. J. Ettlinger, University of Texas.

It is not an uncommon experience of teachers of trigonometry to find students, who are plunged without warning into the definitions of the trigonometric functions, completely bewildered by the terms, sine, cosine, etc. It may be a good plan in taking a cold bath to immerse suddenly, but it is equally a good plan in teaching trigonometry to lead the student into the subject by a method which will directly connect it with his previous study of plane geometry.

This summer in a course in trigonometry the writer developed at some length a graphical introduction, based on the use of the ruler, compasses, and protractor. This graphical substratum continued throughout the course as an essential element.

As a student, the writer recalls a frequent remark of one of his instructors, Professor Roever of Washington University, that the element of value contained in an accurately drawn figure is usually completely overlooked. The habit of drawing by means of instruments careful and accurate figures to represent mathematical situations should be developed as early as possible. Often such a figure suggests a solution; always it will provide a check. It serves admirably as an example of what modern mathematical rigor means, viz: clearness.

The writer puts forward no claim to originality in this paper. A number of the ideas are to be found in the early chapters of Wilczynski and Slaught's *Plane Trigonometry* (Allyn and Bacon). In the course as given the past summer,

however, the text was supplemented by further development and additional problems.

The course opens with a review of plane geometry, covering the theorem of Pythagoras, similar triangles, and the construction of triangles. These theorems are treated in detail. The special case of similar right triangles is given careful attention. The construction of oblique triangles for the four cases is recalled. After pointing out the impossibility of measuring certain distances directly and the magnitude of error introduced when the measurement is possible, we proceed immediately to the graphical solution of triangles.

The use of a ruler and compasses does not require explanation. For this work a ruler graduated to tenths of an inch is a distinct advantage, since with a little practice and the use of fine lines the student can soon estimate accurately to hundredths of an inch. A centimeter scale graduated to millimeters has its merits also.

The rotation definition of an angle is now introduced and defined in terms of the arc subtended on a unit circle. For many purposes it is convenient to make the angle and this arc synonymous. For instance, it removes at once the mystery connected with the notation $\theta = \text{arc sin } y$. It also simplifies the radian unit of measurement. The use of a protractor to measure any angle should be explained.

The student now accurately represents the actual situation in space on a sheet of paper. This he accomplishes by constructing a similar triangle to a suitable scale, and from his knowledge of similar triangles he readily sees why he obtains the correct values for the unknown sides and angles from the figure by direct measurement. For the solution of right triangles, cross-section paper is of great assistance.

The area of any triangle is found by dropping a perpendicular from a vertex to the opposite side as base and measuring the length of the altitude thus obtained. The area is then computed by the usual formula.

The construction of an accurate figure and the careful measurement of the unknown parts of the triangle continues throughout the course in combination with a solution by computation. Later in the course, after the theory of logarithms has been explained and logarithms used in computation, the slide rule is presented as a graphical table of logarithms. It is the belief of the writer that the graphical introduction facilitated the understanding of the principle of the slide rule.

The transition from the graphical method to the use of trigonometric functions is easily made by pointing out the limitations of accuracy to the above constructions. Triangles having small angles and situations demanding more accurate results can be cited. The need for a computation method is appreciated. A course given without this kind of an introduction must necessarily plunge the student into the definition of the trigonometric ratios similar to the arbitrary definitions of the theory of functions. This point of view a freshman cannot understand.

After the trigonometric functions have been defined, the class constructs a

two-place table of sines and cosines by drawing the angles and measuring the lengths. The graph of the sine curve, and cosine also, is drawn, using the data just obtained. This establishes at once the connection between sines and cosines and waves.

The construction of an angle when one of the ratios is given is a useful application of the graphical method. The angle is measured by means of the protractor, and the other ratios are scaled off. Squared paper is very helpful for this work. Finally, it should be pointed out that at this time the inverse trigonometric functions should be introduced as nomenclature for the angles just constructed, and not, as is usually done, at the very end of the course.

It occurs to the writer that some teachers might contend that in a short course in plane trigonometry there is no time to give to the considerations discussed in this paper. In reply the writer can only cite his experience of this summer. The course covered twenty-eight lecture periods, each fifty minutes in length. Out of these, four were devoted to hour quizzes, and two more, to review at the end of the course. In addition to the graphical introduction, all the topics of the average text on plane trigonometry were covered, including proofs for all the formulæ used, trigonometric identities, and trigonometric equations. The only curtailment necessary was in the time ordinarily devoted to the last topic.

From the point of view of interest it may be stated that this introduction makes a strong appeal to all classes: the future engineer always on the alert for practical methods, the bright student studying mathematics for its own sake, as well as the "prerequisite for the A.B." individual. The method, moreover, is valuable *per se* inasmuch as it is the one used during the late war to solve problems in aerial navigation, artillery and orientation, and plane sailing where first approximations were desired. These fields alone provide a wealth of simple problems.

RECENT PUBLICATIONS.

REVIEWS.

Mathematical Essays and Addresses.

The Human Worth of Rigorous Thinking: Essays and Addresses. By Cassius J. Keyser. New York, Columbia University Press, 1916. 314 pp. Price $1.75.

Whitehead and Russell's imposing *Principia Mathematica* furnishes a systematic treatise of the most recent and horoughgoing philosophy of mathematics and as such is addressed to those well grounded in mathematics and logic; Shaw's smaller book, *The Philosophy of Mathematics*, presents the various aspects of mathematics to graduate students. A pleasant task has been assigned to the reviewer, that of describing and evaluating Professor Keyser's delightful essays

and addresses in which he interprets to what are largely non-mathematical audiences the inner spirit of mathematics. Only a mathematician can appreciate the difficulty of his undertaking, and only one who reads or listens to these may know the pleasantness and the effectiveness of the interpretation. These essays are in part known to many through their appearance in such widespread journals as *Science*, *The Hibbert Journal*, and *The Journal of Philosophy, Psychology and Scientific Methods*.

The style in which these essays are rendered must be at once the satisfaction and the despair of all mathematicians—the satisfaction because the subject has so able an exponent, the despair because so very few indeed—let us say one-half of one per cent.!—of the profession have had their tongues touched by so live a coal from the altar.

Strictly speaking, the title applies only to the first three addresses: "The human worth of rigorous thinking," "The human significance of mathematics," "The humanization of the teaching of mathematics," and to the closing address "Mathematics." Pointing out in his first address that mathematics may be characterized by its aim of thinking rigorously whatever is, or may become, rigorously thinkable, or by the collected results of this rigorous thinking, he examines "the just claims to human regard of perfect thought and the spirit of perfect thinking." To comprehend this fully, one must have come to know what rigorous thinking is through a close study of some one or more distinctively mathematical treatises, of the rôle of rigorous thinking in the history of civilization, and of the numberless applications of mathematics to engineering and the natural sciences. This last study makes it evident that "all thinkers and especially students of natural science are engaged, both consciously and unconsciously, both intentionally and unintentionally, in the mathematization of concepts—that is to say, in so transforming and refining concepts as to fit them finally for the amenities of logic and the austerities of rigorous thinking." As examples of this are adduced the concepts of continuity, function, and infinity. At this phase of the development of the subject he takes issue with Bergson and William James in their attack on the method of concepts, the modern method of science, contending that instead of the aim and ideal of intellect being the understanding and subjugation of matter, the essential function of concepts is something more than this, viz., to quote Diotima, "I am persuaded that all men do all things, and the better they are, the better they do them, in the hope of the glorious fame of immortal virtue." He instances further the great difficulty or utter impossibility which these men have on psychological grounds of explaining such contradictions as those, for example, involved in bridging over the transition from sensations, finite in number, to concepts, infinite in multiplicity. What the intellect has done is rather to create for itself a world of conceptual magnitudes which is free from the contradictions of the world of the senses; these magnitudes form the subject-matter of science. The aim of this creative intellect is to preserve and to promote the life of the intellect itself, to think in accordance with the laws of its being, to bring all its laws and methods into complete harmony. Thus "science

and especially mathematics, the ideal form of science, are creations of the intellect in its quest of harmony."

The second address, delivered at the University of California in 1915 before the American Association for the Advancement of Science and affiliated organizations, is too well known to demand a full analysis. Various approaches to the study of the human significance of mathematics are suggested, the historical, the utilitarian, the logical, and the spiritual, the last furnishing the theme for the address, viz., the relation of mathematics to the supreme ideals of mankind. The sovereign passion of mankind is for release from life's limitations and the tyranny of change. Our human aspirations find their highest unity in the quest of a stable world. This quest is to be made, not in the world of sense, but in that of concepts. When on this road toward perfection thought has attained a high degree of refinement, precision, and rigor, we call it mathematical thought. Yet all thought, mathematical or non-mathematical, refined or crude, possesses the unity of being human. Along with the great contributions of theology, philosophy, jurisprudence, art and science to the wealth of the world's knowledge, mathematics has shown to the world that "there exists a stable world of pure thought, a divinely ordered world of ideas, accessible to man, free from the mad dance of time, infinite and eternal." At the risk of evoking a disclaimer from Professor Keyser, one may fairly say that in his protest against the limitation of the motivation of the intellect to a superiority over the material he shares much in common with the ideals of the pragmatists,[1] while his theory of science, and of mathematics in particular, is almost identical with that of Poincaré.

The thesis of the third address, more briefly summarized, is that hope of improvement in the teaching of mathematics, in secondary schools and colleges alike, lies in the possibility of humanizing it, in recognizing that it is not sufficient to say that mathematics possesses its high position as a great human enterprise because it has given the world a metrical and computatory art essential to the conduct of daily life, furnishes countless applications to engineering and the natural sciences, and is an excellent means of giving mental discipline. It is not sufficient even to say with the mathematician that mathematics is the science of pure thought. Rather is it to be said and to be shown in our teaching that its human significance penetrates all fields, it lives in all the activities of men and of nature.

The next four articles, the thirteenth, and the last may be classed together, treating as they do such general mathematical notions as the dimensions of the universe, hyperspace and infinity. Advisedly avoiding the philosophical questions as to the nature of space and accepting it merely as "a vast region or room around us, an immense eternity, locus of all suspended or floating objects of outer sense," the speaker in "The walls of the world" attempts a bold task in seeking to interpret even to an audience of scientists Pascal's characterization of space: "Space is an infinite sphere whose center is everywhere and whose surface is nowhere." A grasp of the notions of infinite sequences and of the equality of

[1] R. B. Perry, *Present Philosophical Tendencies*, third impression, p. 268.

two such sequences, such as would be sufficient for the comprehension of Pascal's statement, is doubtless not to be gained from a single hearing; fortunate is it that the address is available in a form which will allow those interested to reflect more carefully on the notions involved. While Lucretius's argument for the infinite extent of the universe is in a clear fashion shown to depend on a confusion of the terms "boundless" and "infinite," the suggestion is made through a reference to infinities of higher order that our reason may enable us to go in thought beyond the confines even of our infinite universe.

The next article, "Mathematical emancipations" (1906), and the thirteenth, "Concerning multiple interpretations of postulate systems and the 'existence' of hyperspace" (1913), offer a more scientific treatment of the same subject as the foregoing. The former defines "dimension" with due accuracy and develops instances of magnitudes of successively increasing dimensionality, leading up to the fact that ordinary space is not inherently and uniquely three-dimensional. For example, the plane is a three-dimensional space of circles since each circle is determined by three parameters; and on the other hand ordinary space is by the same test a four-dimensional space of lines or of spheres. This interpretation of hyperspace, which is of course essentially that familiar to mathematicians, has the merit of being simple, of making it unnecessary to pass beyond the bounds of the universe or to transcend the limits of intuition. Hyperspace of points however exists not for imagination or intuition, but only for thought and reason. The question of imagining configurations in a hyperspace of points is a question of psychology, not of mathematics, but it can be readily shown that the structure of a fourfold figure can be traced out by its significant analogy with a three-dimensional figure.[1] On the score of accuracy it should be remarked that "Jove nods" when color is named as an example of three-dimensionality in its dependence on the three primary colors; this is true only in the sense of homogeneous coördinates, for a given color depends manifestly only on the two independent ratios of these three components.

The thirteenth article considers the comparative advantages of the languages of geometry and analysis, and argues that n-dimensional geometry has every kind of existence that may be attributed to ordinary geometry of space.

To the assumptions of modern science that the universe is an organic system of order and that it is the sole system of law and order, "The universe and beyond" opposes the view that above every nature there is a supernatural, a hypercosmic. In support of this view it is noted that while mathematics may be defined as

[1] In the presentation of this subject the reviewer has commonly advanced from the straight line to the square, and from this to the cube, pointing out that in our picture of the cube right angles and lines of equal length do not always appear as such; it is then noted that if a hypercube of four dimensions is to be constructed, one would erect a cube on each of the six faces of this cube extending into the fourth dimension, these cubes being "adjacent" in sets of three, and the "outer vertices," eight in number, would determine the last of the eight cubes which bound the hypercube; the number of edges, faces (squares), and vertices are enumerated by analogy with two and three dimensions, and finally *the picture of the hypercube* (the projection into three-dimensional space) *is given*, this comprising a cube entirely outside a second cube with the pairs of corresponding vertices joined by eight lines.

the science of measurement, direct or indirect, and of position, as the science of operations, etc., as characterized by its content, we may also define it by its method as in the well-known definition of Benjamin Peirce; and since pure mathematics is intrinsically concerned with accuracy, correctness and completeness in its logic rather than with the truth of its applications to our universe, which is the domain of natural science, it may conceivably deal with that which transcends the sensible universe, as when for example one deals in euclidean and non-euclidean geometries with inconsistent systems of consistent relationships.

The last article, "Mathematics," supplements the characterization of mathematics as given in the preceding paper by a presentation of the growth of mathematics as a science through the critical movement grounded in the study of the theory of functions and of the foundations and in a study of symbolic logic which discloses the basis of logic as identical with the basis of mathematics. The background of Keyser's writing seems clearly to be the philosophy of mathematics embodied in Whitehead and Russell's *Principia Mathematica*, the first two volumes of which he has reviewed in the twelfth chapter of the present collection. (The reader who desires to follow the discussion intensively needs to refer for Russell's latest interpretations to his recently published *Introduction to Mathematical Philosophy*.) Hamilton's and Schopenhauer's indictments of mathematics are rejected by Keyser as consisting of malicious misrepresentation; those of Huxley as based on actual ignorance of the nature of mathematical thinking in its relation to observation and causation. It is not surprising that a writer so zealous for the purity and for the spiritual values of mathematics should in the closing part of this paper disparage the term "applied mathematics" and distinguish sharply between mathematics and natural science. This indicates a divergence, partly of emphasis, partly of definition, from the usage of the many who view mathematics as inclusive of its applications.

Incidentally taking issue with the views of Professor Royce as to the possibility and validity of existence proofs of the infinite, the seventh article considers "The axiom of infinity." Mankind by the need of considering the indefinitely small and the indefinitely large has been forced through mathematics, as through ethics and philosophy, to treat the infinite. This search has brought high rewards, even if it were found to be an insoluble problem. The studies of Riemann, Bolzano, Dedekind and Cantor are suggested as having for the first time in history obtained a logical hold on infinity. The contention is rightly made that when some modern critics characterize Bolzano's definition of infinity based on the inexhaustibility of a class as a negative definition, or of the infinite as the really positive and the finite as the negative, the distinction is not essential but represents merely an arbitrarily chosen point of view. Yet, to the reviewer's mind, Professor Keyser is guilty of this very "verbal legerdemain" when he insists that Royce, Russell and others are wrong in saying that the concept of the infinite denies the axiom of the whole and part because the common-sense axiom is applicable only when there is a number telling now many; this contention stands or falls solely with the particular definition given to the terms involved.

The most interesting portion of this essay is the argument[1] that it is impossible to prove the existence of the infinite, that the demonstrations of such existence involve reasoning in a circle, and that one must instead adopt the axiom of infinity which states that "conception and logical inference alike presuppose absolute certainty that an act which the mind finds itself capable of performing is intrinsically performable endlessly, or, what is the same thing, that the assemblage of possible repetitions of a once performable act is equivalent to some proper part of the assemblage."

The remaining articles are contributions to educational discussions. "The permanent basis of a liberal education" points out that amid the unceasingly changing state of knowledge there are certain great invariant facts which should be enshrined in any plan providing a liberal education; such facts are the immense past of mankind, the material universe in which we find ourselves, our dependence on the world of ideas, the social character of our world, the fact of the human future, and the discipline of beauty.

The address on "Graduate mathematical instruction . . ." calls upon university departments to offer such courses as will enable students to gain a general knowledge of the problems and methods of these fields even though specializing in other departments. It is maintained on the ground of experience and observation that there is an opportunity for such courses and a distinct duty of the universities in the matter, in spite of the fact that the intrinsic technicality of each subject may make it difficult and in part impossible for the instructor to make it known to the non-specialist. The reviewer may in this connection instance the philosopher Josiah Royce, who in the later part of his life with the wealth of his intellectual equipment studied the fundamentals of modern mathematics with the double effect of showing in his writings an unusually intelligent and an unusually sympathetic grasp of mathematics in its philosophical bearings. How happy a contribution might be made to a clean-cut organizing and harmonizing of the various systems of philosophy if only more frequently mathematicians would bring to that study the instrumentality of their critico-logical training! Keyser would demand as a prerequisite only a year of collegiate mathematics, since in his view the material most valuable for such avocational instruction does not depend on the calculus. The course would comprise a presentation of the nature of mathematics, of postulate systems and the rôle they have played in ancient and modern times, the function concept, the limit concept, the calculus, the theory of point sets, invariants, and groups.

Three other chapters are of more general import, one of which, "Mathematical productivity in the United States" (1902) might wisely have been supplemented by the encouraging contrast which a summary of present conditions would furnish.

W. D. CAIRNS.

[1] C. J. Keyser, "The axiom of infinity and mathematical induction," *Bulletin of the American Mathematical Society*, Vol. 9, May, 1903.

NOTE ON PRIME NUMBERS.

In my review of the first volume of Dickson's *History of the Theory of Numbers* in this MONTHLY (*1919*, 402), I quote from page 425 of the *History* the conjecture of M. Cantor (correctly reproduced from his paper) that three successive primes are not in arithmetical progression unless one of them is 3. Through the kindness of Professor F. H. Loud my attention has been called to the fact that the conjectured theorem is false. In fact from a table of primes it is easy to find sets of three consecutive primes with common difference 6, for example. The four primes 251, 257, 263, 269 are successive and are in arithmetical progression.

R. D. CARMICHAEL.

Plane Trigonometry and Numerical Computation. By J. W. YOUNG and F. M. MORGAN. New York, Macmillan, 1919. 12mo. 7 + 122 pp. Price $1.25.

The Preface: "Ever since the publication of our *Elementary Mathematical Analysis* (The Macmillan Co., 1917) we have been asked by numerous teachers to publish separately, as a textbook in plane trigonometry, the material on trigonometry and logarithms of the text mentioned.

"The present textbook is the direct outcome of these requests. Of course, such separate publication of material taken out of the body of another book necessitated some changes and an introductory chapter. As a matter of fact, however, we have found it desirable to make a number of changes and additions not required by the necessities of separate publication. As a result fully half of the material has been entirely rewritten, with the purpose of bringing the text abreast of the most recent tendencies in the teaching of trigonometry.

"There is an increasing demand for a brief text emphasizing the numerical aspect of trigonometry and giving only so much of the theory as is necessary for a thorough understanding of the numerical applications. The material has therefore been arranged in such a way that the first six chapters give the essentials of a course in numerical trigonometry and logarithmic computation. The remainder of the theory usually given in the longer courses is contained in the last two chapters.

"More emphasis than hitherto has been placed on the use of tables. For this reason a table of squares and square roots has been added. Recent experience has emphasized the applications of trigonometry in navigation. We have accordingly added some material in the text on navigation, have introduced the haversine, and have added a four-place table of haversines for the benefit of those teachers who feel that the use of the haversine in the solution of triangles is desirable. The material can, however, be readily omitted by any teacher who prefers to do so."

NOTES.

The fifth quinquennial *Table des matières* of *Revue semestrielle des publications mathématiques* was published in 1918. It covers volumes 21–25 (1913–1917).

Noordhoff of Groningen published in 1918 the second volume of the *Oeuvres complètes de Thomas Jan Stieltjes* which are being issued under the auspices of the Amsterdam mathematical society. It is a quarto volume of over six hundred pages. The first volume, which was about one-third smaller, appeared in 1914.

The fourth edition of Appell's *Traité de mécanique rationnelle*, tome 1, was published in 1919. It contains about five pages more than the third edition.—Recent publications by R. de Montessus de Ballore are: *Introduction à la théorie des courbes gauches algébriques* (Paris, Croville-Morant, 1918); the sixth enlarged and thoroughly revised edition of the elementary algebra part of C. de Comberousse's *Cours de mathématiques* (Paris, Gauthier-Villars, 1919); and the *Yearbook of the Universities* to which we have already made reference (*1919*, 358).

Recent catalogues of second hand mathematical books: No. 97 (495 titles in mathematics and physics), Galloway and Porter, Cambridge, England; No. 773 (3336 titles in exact and applied science), Henry Sotheran and Co., 140 Strand, W. C. 2, London. The latter catalogue includes the library of Olaus Henrici.

Among the articles in *Revista matemática Hispano-Americana*, September, 1919 are: "El tratado de la logarítmica de Juan Caramuel" by D. F. DIÉGUEZ pp. 203–212 (including facsimiles of two pages of tables); and 'La teoría de los grupos en la enciclopedia matemática" by G. A. MILLER, pp. 229–232. Cf. *1919*, 302.

The frontispiece of *Popular Astronomy* for November, 1919 is a group picture of members and visitors at the Ann Arbor meeting of the American Astronomical Society, September 2–5. Among those in the group are the following members of the Association: E. W. Brown, W. D. MacMillan, F. R. Moulton, and E. D. Roe, Jr. Elsewhere in the issue is a fine portrait of Benjamin Baillaud (director of the Paris observatory, and first president of the International Astronomical Union), and a picture of the Palais des Académies where the meetings of the International Research Council were held in Brussels last July.

Science, for December 19, records that at a recent meeting of the corporation of Yale University it was voted "to extend the sincere congratulations of the corporation to Professor ERNEST BROWN on the completion of his monumental work on the *Tables of the Motion of the Moon*, and to assure him that the university considers the work that he has done on these volumes as among the most important scientific contributions ever made by an officer of Yale University."

G. Bell and Sons, London, have announced the publication of *Nomography* by S. BRODETSKY, and of *Differential Equations and their Applications* by H. PIOGGIO.— Dunod, Paris, has published a French translation, by A. E Gérard, of SILVANUS P. THOMPSON'S *Calculus Made Easy* (first published anonymously in 1910) with the title: *Le calcul intégral et différentiel à la portée de tout le monde.*

Engineering Education. Essays for English. Selected and edited by R. P. BAKER (New York, Wiley, 1919) includes the following: Under the heading The Origins of Engineering Education, "Evolution of the scientific investigator" by S. Newcomb, pages 3–28 (reprinted from the Smithsonian Institution *Report*, 1904); under the heading The Bases of Engineering Education—Mathematics, "The place of mathematics in engineering practice" by W. H. White, pages 103–112 (reprinted from *Nature*, September 19, 1912), and "On the relation of mathematics to engineering" by A. Ranum, pages 113–121 (reprinted from the *Sibley Journal of Engineering*, January, 1914). Each essay is preceded by a brief biographical sketch of the author.

The following paragraph appeared in *Nature* for November 27:

"The unfailing energy of Prof. Pearson's department at University College, London, has now resulted in the production of a series of tracts published by the Cambridge University Press.

The objects of this new series are not only to publish new tables (as well as to republish old and inaccessible tables), but also in due course to issue works on interpolation, mechanical quadratures, calculating machines, and other matters of importance to the practical computer. The first of the series is before us, and is entitled *Tables of the Digamma and Trigamma Functions*, by ELEANOR PAIRMAN. The work contains tables of the logarithmic derivate of the Gaussian Π-function and of its derivate, in addition to some useful miscellaneous information concerning these two functions. The functions are tabulated to eight places of decimals at intervals of 0.02 from 0 to 16, with tables of second differences. There seems no doubt that this series will be of extreme value to computers, and we must feel deep gratitude to Prof. Pearson for using the resources at his disposal in producing it. Finally, it should be said that the appearance of the first of the series is up to the standard which we have grown accustomed to expect from the Cambridge University Press."

ARTICLES IN CURRENT PERIODICALS.

ANNALS OF MATHEMATICS, 2d series, volume 21, no. 1, September, 1919: "Investigation of a class of fundamental inequalities in the theory of analytic functions" by J. L. W. V. Jensen [translation from the Danish by T. H. Gronwall], 1–29; "Functions of limited variation in an infinite number of dimensions" by P. J. Daniell, 30–38; "A new sequence of integral tests for the convergence and divergence of infinite series" by R. W. Brink, 39–60; "Calculation of the complex zeros of the function $P(z)$ complementary to the incomplete gamma function" by P. Franklin, 61–63; "Total differentiability" by E. J. Townsend, 64–72.

ATHENAEUM, 1919, November 14: "Einstein's theory of gravitation" by X., 1189 [Last paragraphs: "Einstein supposes that space is euclidean where it is sufficiently remote from matter, but that the presence of matter causes it to become slightly non-euclidean—the more matter there is in the neighbourhood, the more space will depart from Euclid. By the help of this hypothesis, together with his previous theory of relativity, he deduces gravitation—very approximately, but not exactly, according to the Newtonian law of the inverse square.

"The minute differences between the effects deduced from his theory and those deduced from Newton are measurable in certain cases. There are, so far, three crucial tests of the relative accuracy of the new theory and the old.

"(1) The perihelion of Mercury shows a discrepancy which has long puzzled astronomers. This discrepancy is fully accounted for by Einstein. At the time when he published his theory, this was its only experimental verification.

"(2) Modern physicists were willing to suppose that light might be subject to gravitation, *i.e.*, that a ray of light passing near a great mass like the sun might be deflected to the extent to which a particle moving with the same velocity would be deflected according to the orthodox theory of gravitation. But Einstein's theory required that the light should be deflected just twice as much as this. The matter could only be tested during an eclipse among a number of bright stars. Fortunately a peculiarly favourable eclipse occurred this year. The results of the observations have now been published, and are found to verify Einstein's prediction. The verification is not, of course, quite exact; with such delicate observations that was not to be expected. In some cases the departure is considerable. But taking the average of the best series of observations, the deflection at the sun's limb is found to be 1.98″, with a probable error of about 6 per cent., whereas the deflection calculated by Einstein's theory should be 1.75″. It will be noticed that Einstein's theory gave a deflection twice as large as that predicted by the orthodox theory, and that the observed deflection is slightly *larger* than Einstein predicted. The discrepancy is well within what might be expected in view of the minuteness of the measurements. It is therefore generally acknowledged by astronomers that the outcome is a triumph for Einstein.

"(3) In the excitement of this sensational verification, there has been a tendency to overlook the third experimental test to which Einstein's theory was to be subjected. If his theory is correct as it stands, there ought, in a gravitational field, to be a displacement of the lines of the spectrum towards the red. No such effect has been discovered. Spectroscopists maintain that, so far as can be seen at present, there is no way of accounting for this failure if Einstein's theory in its present form is assumed. They admit that some compensating cause *may* be discovered to explain the discrepancy, but they think it far more probable that Einstein's theory requires some essential modification. Meanwhile, a certain suspense of judgment is called for. The new law has been so amazingly successful in two of the three tests that there must be something valid about it, even if it is not exactly right as yet.

"Einstein's theory has the very highest degree of aesthetic merit: every lover of the beautiful must wish it to be true. It gives a vast unified survey of the operations of nature, with a technical simplicity in the critical assumptions which makes the wealth of deductions astonishing. It is a case of an advance arrived at by pure theory: the whole effect of Einstein's work is to make physics more philosophical (in a good sense), and to restore some of that intellectual unity which belonged to the great scientific systems of the seventeenth and eighteenth centuries, but which was lost through increasing specialization and the overwhelming mass of detailed knowledge. In some ways our age is not a good one to live in, but for those who are interested in physics there are great compensations."]

JOURNAL OF THE INDIAN MATHEMATICAL SOCIETY, volume 11, no. 4, August, 1919: "Progress report" by D. D. Kapadia, 121; "Mr. S. Ramanujan, B.A., F.R.S.," 122 and portrait frontispiece ["Was born of poor parents in December, 1888, at Erode in the Madras Presidency . . . In 1910 he went to Madras with two good sized note books filled with his researches in mathematics and with the help of some friends obtained a clerk's post in the Harbour Trust Office. This enabled him to remain in Madras and work at mathematics with the help of books and periodicals which were made accessible to him there. In 1912, having one day seen the remark in Mr. G. H. Hardy's tract on *Orders of Infinity* that the precise order $\rho(x)$ had not yet been ascertained and having himself arrived at a result relating to it, he opened correspondence with Mr. Hardy sending him some of his results on continued fractions and theory of numbers. Mr. Hardy was struck with the grandeur of the results and wrote to him an appreciative letter asking for more of his results in other branches of mathematics. On receiving these which were mainly in definite integrals and elliptic functions, Mr. Hardy discovered in Ramanujan a great mathematician and asked him if he would go to Cambridge . . . The University of Madras granted Mr. Ramanujan a special scholarship of the annual value of £250 for three years and he sailed for England in March, 1914 . . . An account of his work while in England has already appeared in the February, 1917 number of this *Journal*. . . . In recognition of his abilities in mathematics he was elected a fellow of the Royal Society in 1917, and in 1918 he was granted a Fellowship of Trinity. The University of Madras has also been pleased to give him an annual grant of £250 for another period of five years without imposing any conditions regarding residence or work. Since May 1917 he has not been keeping good health. He returned to India in March last for the sake of his health and is now residing . . . very near the place of his birth"]; "On the cartesian oval" by V. R. Aiyar, 123–144; "Some difficulties met with in reading mathematics without a teacher: a complaint against text-books" by W. A. Garstin, 145–154; "Astronomical notes" by T. P. B. Sastri, 155; Problems and Solutions, 156–160; "Mathematics (in brief) from current periodicals," i–iii.

MATHEMATICS TEACHER, volume 12, no. 1, September, 1919: "An experiment in motivation" by W. S. Schlauch, 1–9; "Scales for the study of children's characteristics" by E. S. Smith, 10–16; "Applied mathematics in high schools: some lessons from the war" by W. E. Breckenridge, 17–22; "A junior high school course in mathematics" by Emily Renshaw, 23–27; "A simple method of reconstructing a hyperbolic paraboloid" by E. J. Cuy (Couyundjopoulos), 28–29; "The national committee on mathematical requirements," 30–32; Editorials, Book Reviews, Notes and News, 33–40.

MESSENGER OF MATHEMATICS, volume 48, no. 9, January, 1919: "Note on the deflection of beams" by W. H. Macaulay, 129–130; "Laws of facility of error" by A. R. Forsyth, 131–144—No. 10, February: "On Napier's circular parts" by W. W. Johnson, 145–153 [Historical]; "Theorems in the expansion of polynomials, obtained by an application of the calculus of residues" by E. A. Milne, 153–159; "The dissection of rectilineal figures" by W. H. Macaulay, 159–160.

NOUVELLES ANNALES DE MATHÉMATIQUES, volume 78, June, 1919: "Réduction à une forme normale d'un système d'équations différentielles simultanées linéaires à coefficients constants" by H. Vogt, 201–209; "Le théorème de Feuerbach dans les cubiques" by Malgouzon, 210–213; "Démonstration du théorème de Chasles sur les arcs égaux de lemniscate" by F. Balitrand, 213–215; "Sur les conditions pour qu'une fonction $P(x, y) + iQ(x, y)$ soit monogène" by M. Fréchet, 215–219; "Groupes de points sur l'hyperbole équilatère" by J. Ser, 220–228; Questions and solutions, 230–240.

PENNSYLVANIA SCHOOL JOURNAL, volume 67, April, 1919: "The Courtis tests in arithmetic" by W. A. Boyer, 480–481.

PROCEEDINGS OF THE AMERICAN PHILOSOPHICAL SOCIETY, volume 58, 1919: "Graphical representation of functions of the Nth degree" by F. E. Nipher, 236–240.

Revue de Métaphysique et de Morale, volume 26, no. 5, September–October, 1919: Etudes critiques—"Les Principes de l'analyse mathématique par Pierre Boutroux" by M. Winter, 649–667 [Last sentences: "Le livre que nous venons d'analyser donnera au jeune géomètre une vue d'ensemble de la science à laquelle il travaille. L'ouvrage de M. Boutroux intéressera également le philosophe; les grands courants de la pensée qui se dissimulent sous les théories mathématiques y sont discrètement décrits: l'auteur nous signale les heurts, les chocs qui en resultent. Il n'a pas cru devoir rester 'au-dessus de la mêlée'. Il a pris parti, il a défendu les idées qui nous semblent les plus conformes au développement de la science."]

Revue de Philosophie, année 19, no. 4: July–August, 1919: "Mathématique et métaphysique" by P. M. Périer, 384–395 (to be continued); "Pierre Duhem (1861–1916)", 457–462 [Quotations: "M. Duhem Pierre-Maurice-Marie naquit à Paris le 10 juin 1861. Elève du collège Stanislas, il se voue à l'étude des sciences et se prépare à l'Ecole Normale Supérieure. Il y est admis à l'âge de vingt et un ans. Son intelligence lucide, son travail obstiné, le font bientôt classer parmi les meilleurs sujets. Après ses trois années d'études, le nouvel agrégé, sorti premier du concours, aurait pu avoir hâte de jouir de la situation qu'il s'etait créée. Les deux années suivantes, il demanda et obtint la faveur de rester à l'Ecole en vue de perfectionner sa formation scientifique. . . . Peu d'hommes cependant ont été de tous points plus meritants que M. Duhem. Magnifiquement doué du point de vue intellectuel, à la fois *historien*, *philosophe* et surtout *savant*, il sut allier à la science abstraite des dispositions artistiques, qui ne laissaient pas de surprendre dans l'homme du chiffre et de l'abstraction. Partout où il s'offrait à lui, dans l'art, en littérature, dans l'architecture ou dans la nature, le beau ne le trouva jamais indifférent. Sa faculté esthétique n'explique-t-elle pas le charme singulier et la coloration de sa parole et surtout de sa claire composition?"].

Revue Philosophique de la France et de l'Etranger, année 44, September–October, 1919: "L'imagination pure et la pensée scientifique" by J. Segond, 297–321.

Revue scientifique, année 57, no. 17, 6–13 September, 1919: "Le temps et sa mesure" (suite) by M. Moulin, 517–527.

School Review, volume 27, no. 8, October, 1919: "Geometry by analysis" by H. O. Barnes, 612–618.

School and Society, volume 10, August 23, 1919: "What and how far have military courses and training contributed to the college curricula?" by P. P. Boyd, 219–224—October 11: "The National Committee on Mathematical Requirements," 435–437.

School Science and Mathematics, volume 19, no. 8, November, 1919: "A practical method for demonstrating the error of mean square" by H. F. Roberts, 677–692 [bibliography on page 692]; "How x came to stand for unknown quantity" by F. Cajori, 698–699 [Quotation: "As a matter of fact, there is no evidence, worthy of serious consideration, to show that x was used as the symbol for unknown quantity before the publication of Descartes' *Géométrie*, in 1637"]; "A 'flu' dream in mathematics" by W. A. Austin, 701–713; "Horner's method" by Elizabeth Sanford, 726 [First of five verses:

"Have you ever asked why asylums were filled
With people whose sweet dispositions were killed?
It's because of one man, with a Mother Goose name,
Who invented a system to drive folks insane.
That's Horner."];

"Force, work and power—their relation. Note on relation of work to heat added" by S. A. Garlick, 727–731; "Nineteenth meeting of the Central Association of science and mathematics teachers," 751; Problems and solutions, 755–758; "New outline map of the United States on the Lambert projection," 760; "The National Committee on Mathematical Requirements," 763–764; Review by G. A. Miller of Cajori's *History of Mathematics* (new edition, New York, 1919), 768, 770.

Science, new series, volume 50, November 7, 1919: Review by H. S. White of T. Smith and R. W. Cheshire's *Constructional Data for Small Telescope Objectives* and *Additional Data, etc.* (London, 1915–1916), 437–439—November 21: "The history of science and the American Historical Association" by E. J. Benton, 478; "The deflection of light by gravitation and the theory of relativity," 478–479—November 28: "The historical point of view in the teaching of science" by G. A. Miller, 489–493 [Address before the Missouri State Teachers Association. First two paragraphs: "The teachers of Missouri should take special interest in the history of science

at the present time in view of the fact that the American Association for the Advancement of Science is expected to meet soon in this state and the question of forming a special section of this association for the purpose of considering topics in the history of science is to be raised during this meeting. Teachers of mathematics have an additional reason for taking an unusually keen interest in this subject just now in view of the appearance during the past summer of two very important books on the history of their subject.

"One of these is entitled 'History of the Theory of Numbers' and was prepared by Professor L. E. Dickson of the University of Chicago, while the other bears the more general title 'A History of Mathematics' and was prepared by Professor Florian Cajori, of the University of California, who holds the unique position of a regular professorship of the history of mathematics in a university. The former book is the first volume of the most complete history of number theory ever written and marks an epoch in American mathematical literature, while the latter is technically only a 'revised and enlarged edition' of a book which appeared a quarter of a century ago under the same title, but the changes are so extensive that it too may be regarded as practically a new work."]

SCIENTIFIC MONTHLY, volume 9, no. 5, November, 1919: "The controversy on the origin of our numerals" by F. Cajori, 458–464. [The article closes with the following summary: "The following are the outstanding facts:

1. The earliest reliable record of the use of our numerals with the zero is an inscription of 867 A.D. *in India.*
2. The validity of the testimony of early Arabic writers ascribing to India the numerals with the zero is shaken, but not destroyed.
3. There is not a scintilla of evidence in the form of old manuscripts or numeral inscriptions to support the Greek origin of our numerals.
4. At present the hypothesis of the Hindu origin of our numerals stands without any serious rival. But this hypothesis is by no means firmly established.

As a by-product of the discussion of recent years we must admit that, on the evidence presented, the claim that our numerals and the zero were used in India as early as the fifth century must be abandoned; our earliest apparently reliable evidence belongs to the ninth century. We must also abandon the claim that the early Hindus used the abacus, the rule of 'double false position,' and the process of 'casting out the nines.' These corrections are due to G. R. Kaye."]

UNIVERSITY OF TEXAS BULLETIN, no. 1904, January, 1919: *A comparison of the premiums of the Teachers Insurance and Annuity Association with those of other legal reserve companies,* by E. L. Dodd, 19 pp.

UNDERGRADUATE MATHEMATICS CLUBS.

EDITED BY U. G. MITCHELL, University of Kansas, Lawrence.

THE MATHEMATICS CLUB OF THE UNIVERSITY OF KANSAS, Lawrence, Kansas.

[*1918*, 35, 450, 459; *1919*, 208.]

The officers of the club for the year 1919–20 are as follows: President, Jessie Craig '20; vice-president, Beatrice Hagen '20; secretary-treasurer, Evelina Watt '20; reporter, Etna Morrison '20; faculty adviser, Professor Ellis B. Stouffer; program committee, the faculty adviser, the secretary, the reporter and Ruth Kelsey '20.

Below are given the programs for the year 1919–20, as announced in the printed folder issued by the club.

October 9, 1919: "Codes and Ciphers" by Professor Ulysses G. Mitchell.
October 23: "The game of nim" by Etna Morrison '20.

November 13: "De Moivre's theorem" by Lucile Noah '20.
December 11: "Hyperbolic functions" by Marie McKinney '21.
January 8, 1920: "Sailing quicker than the wind" by Marie Shaklee '21; "Famous problems of antiquity" by Nadene Weibel '21.
January 22: "Mathematics of the calendar" by Lillie Strand '21.
February 12: "Life and works of Newton" by Hilda Bushnell '21; "The Leibniz-Newton controversy" by Wayne Stevenson '20.
February 26: "Mathematical machines" by Victor Mellenbruch Gr., and Eran O. Burgert Gr.
March 11: "Mathematics in France" by Professor Solomon L. Lefschetz; "Mathematicians of America" by Professor Cyril A. Nelson.
March 25: "Flatland" by Nina McLatchey Gr. Play "Flatlanders" by members of the club.
April 8: "Origin of number symbols" by Marie Power '20.
April 22: "Geometric fallacies" by Marie Brown '21.
May 13: "Magic squares" by Pearl L. Mellenbruch Gr.
May 27: Annual picnic.

THE MATHEMATICAL CLUB OF ROCKFORD COLLEGE, Rockford, Illinois. [*1918*, 188, 409.]

The officers of the club for the year 1918–19 were: President, Ruth Gleasman '19; vice-president, Dorothy Jamison '20; secretary-treasurer, Lucy E. Brown '20.

The following programs were given during the year.
September, 1918: Initiation of new members.
November: "The Ueberfeld Horses," from Maeterlinck's *The Unknown Guest*, by Lea Gordon '20.
December: "Problems in paper cutting" by Aline Bartholomew '20.
January, 1919: "Comet's tails" by Professor Bessie I. Miller.
February: "Magic squares and mathematical tricks" by Dorothy Jamison '20.
March: Geometrical recreations—"Geometrical fallacies" by Anna Foster '21; "Geometrical paradoxes" by Bohnmilla Hrdlicka '21; "Geodesics" by Elizabeth Rearick '20.
April: "Short cuts in calculation" by Margaret Dodd '21.
May: "The theory of probability" by Ruth Gleasman '19.

Membership in the club is limited to students who have taken or are taking elective work in mathematics. At present there are seventeen members. The officers for the year 1919–20 are: President, Dorothy Jamison '20; vice-president, Virginia Schneider '20; secretary-treasurer, Frances Regan '21.

THE UNIVERSITY OF SASKATCHEWAN MATHEMATICAL SOCIETY, Saskatoon, Saskatchewan. [*1918*, 270, 460.]

On account of the interruption of the University work by the influenza epidemic no meeting of the society was held during the year 1918–19 before

Christmas. The first meeting of the new year was held in February, 1919. Gladys Shannon '20 presented a paper on "The concept of infinity in geometry." The paper was followed by considerable discussion and it was decided that the subject should be given further consideration at the second meeting.

The second meeting was held at the residence of Professor Lloyd L. Dines and about twenty members were present. The discussion of the subject of the first meeting was continued by John H. Simester '20, Lillian Williamson '21, Helen Fernald '21 and Gladys Shannon '20, as well as by nearly all of the other members present.

The third and last meeting for the year 1918–19 was held at the residence of Dean George H. Ling and about fifteen members were present. John H. Simester '20 presented a paper on "The logic of algebra" and after a discussion of it several interesting problems were presented by members of the club. A resolution was introduced and passed unanimously that hereafter the society should be called "The Shuttleworth Mathematical Society of the University of Saskatchewan" in honor of the late Roy Shuttleworth, who was the first president of the club and who fell in France in August 1918 after having won the Military Medal for distinguished services.

The following officers were elected for the year 1919–20: President, John H. Simester '20; vice-president, Gladys Shannon '20; secretary-treasurer, Andrew M. Ridout '21.

The Mathematics Club of Vassar College, Poughkeepsie, N. Y.

[*1918*, 136, 456; *1919*, 264.]

The officers for the first semester of the year 1918–19 were: President, Helen Thompson '19; vice-president, Susan Burr '20; secretary-treasurer, Helen Lewis '20; faculty member of executive committee, Professor Elizabeth B. Cowley; student member of executive committee, Mildred Booth '20.

Below are given the programs for the year 1918–19.

October 4, 1918: Business meeting for election of officers for first semester. After the election the members of the club "took a hike."

November 6: "Spherical aberration" by Professor Henry S. White; "Some mathematical fallacies" by Kathleen Millay '21. The members of the club had an animated discussion over the fallacies. Some problems were proposed for solution before the next meeting.

December 3: "Air resistance against rotating projectiles" by Eliza Sommerville '20; "Underwater trajectories and phenomena of flight" by Mildred Werntz '20. It was announced that Amy Davison '21 had solved the problems proposed at the last meeting. It was also announced that a list of examples and questions that were intended to test personal efficiency had been placed on the bulletin board. They had been clipped from a newspaper.

January 16, 1919: A play "Flatlanders"[1] written by Kathleen Millay '21 and Lucille Free '21 was presented by members of the club.

[1] Published in full in the June, 1919, number of the Monthly, pp. 264–267.

February 27: Election of officers for second semester: President, Helen Thompson '19; vice-president, Susan Burr '20; secretary-treasurer, Helen Lewis '20; faculty member of executive committee, Professor Cowley; student member of executive committee, Mary Coye '20.

Subject of program "The fourth dimension." Mary Coye '20 explained it by the use of an analogy; Katherine Jaeger '19 gave historical data; Amy Davison '21 discussed some of the theoretical aspects.

It was recommended that in the future some of the meetings be held in other places than in Rockefeller Hall (recitation hall).

April 10: "A brief account of the life of Lewis Carroll" by Pauline Chrisler '19; "Lewis Carroll as a mathematician" by Christie White '19; "An adaptation of Stephen Leacock's essay on 'A, B, and C'" (reading) by Jean Proutt '19.

The meeting was held in senior parlor. Mrs. H. S. White and some freshmen were invited as guests of the club. After the formal program and business session were over tea was served.

May 20: The club had a picnic near the "old lake." A short business meeting was held and then mathematical games were played until supper time.

The officers elected for the first semester of the year 1919–20 are: President, Mildred Booth '20; vice-president, Amy Davison '21; secretary-treasurer, Susan Burr '20; faculty member of the executive committee, Professor Cowley; student member of the executive committee, Mary Coye '20.

SUMMARY NOTES—1919.

The establishment of the Students Army Training Camps at various universities and the outbreak of the epidemic of influenza in the fall of 1918 interfered greatly with the work of undergraduate mathematics clubs. Of thirty-three clubs which reported during 1919 on the work of the school year 1918–19, nineteen—University of Alabama, University of Chicago, University of Colorado, Greenville College, Grinnell College, University of North Carolina, University of Oregon, Swarthmore College, Albion College, Columbia University, Denison University, Harvard University, University of Illinois, University of Kansas, University of Kentucky, University of Maine and University of Nebraska—do not mention any meetings during the period from September first 1918 to January first 1919 and the first eight of these clubs are reported as remaining inactive during the entire school year. Nearly all of these inactive clubs have reported that they expect their work during the year 1919–20 to be back to normal activity again.

One new club, that of the University of Iowa, was organized in April 1919.

During 1919 the MONTHLY has published reports of the activities of twenty different clubs. These reports included accounts of more than 100 meetings and a list of about 160 topics discussed. Discussions of two club topics, "The Number π" and "Codes and Ciphers," have been published and some suggestions given as to forms of entertainment for social meetings.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND OTTO DUNKEL.

[Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Missouri.]

PROBLEMS FOR SOLUTION.

2806. Proposed by WARREN WEAVER, Throop College of Technology.

The Maxwell law for the distribution of velocities among the molecules of a gas shows that, considering a velocity of given magnitude, it is equally probable that it have any direction. Find an expression for the probability that the velocity of a molecule m_2 (whose velocity relative to the mass motion of the gas is k') when measured relative to the velocity k of a molecule m_1 (k less than k') shall make with k an angle falling within the limits θ and $\theta + d\theta$.

2807. Proposed by S. A. COREY, Des Moines, Iowa.

Establish the identity

$$\begin{vmatrix} -x & bcy & -acu & -abv \\ y & x & -av & au \\ u & bv & x & by \\ -v & cu & cy & -x \end{vmatrix}^2 + ac \begin{vmatrix} x & bcy & -x & -abv \\ -y & x & y & au \\ u & bv & u & by \\ -v & cu & -v & -x \end{vmatrix}^2$$

$$+ ab \begin{vmatrix} x & bcy & -acu & -x \\ -y & x & -av & y \\ u & bv & x & u \\ -v & cu & cy & -v \end{vmatrix}^2 = \begin{vmatrix} x & bcy & -acu & -abv \\ -y & x & -av & au \\ u & bv & x & by \\ -v & cu & cy & -x \end{vmatrix}^2.$$

2808. Proposed by A. A. BENNETT, University of Texas.

Let $f(x)$ denote an arbitrary frequency distribution function.

(1) Show that the condition that frequencies when compounded follow the law for simple frequencies is that the iteration of $f(r-t)$ in the sense of integral equations leaves this kernel unaltered, *i.e.*,

(*i*) $f(r-t) = \int_{-\infty}^{+\infty} f(r-s)f(s-t)ds$ and, hence, this kernel is singular.

Cf. David F. Barrow, *Annals of Mathematics*, second series, vol. 19 (1917), p. 97.

(2) Show that $f(a, x)$ satisfies (*i*) for every real value of a, when

$$f(a, x) = \begin{cases} \dfrac{1}{\pi x} \sin ax, \text{ for } -c < x < c, \text{ } c \text{ being arbitrary, real and positive but fixed. In particular } c \text{ may be infinite.} \\ 0, \begin{cases} \text{for } x < -c. \\ \text{for } x > c. \end{cases} \end{cases}$$

(3) Show that for a given c, and $0 < a < b$, $f(b, x) - f(a, x)$ and $f(a, x)$ are orthogonal.

(4) Show that the Gaussian normal probability function satisfies (*i*). Consider its expansion in terms of the orthogonal functions of (3).

(5) Show that for c finite, $f(a, x)$ cannot be defined by means of a Fourier series, save for discrete values of a, since for a varying within certain restricted segments, the Fourier coefficients of $f(a, x)$ remain unaltered. Show that when a Fourier representation is possible, the series reduces to a trigonometric polynomial.

2809. Proposed by the late L. G. WELD.

Find the nth term of the series defined by the relation, $u_{i+2} = u_i + u_{i+1}$, in which $u_1 = u_2 = 1$.

2810. Proposed by H. S. UHLER, Yale University.

In the expansion of the following determinant or eliminant, find the total number of terms, and the number of terms having the coefficients $+1, -1, +2, -2, +3, -3, +4, -4, +5, -5, +6, -8, +10$, respectively.

$$\begin{vmatrix} a & b & c & d & e & 0 & 0 & 0 \\ 0 & a & b & c & d & e & 0 & 0 \\ 0 & 0 & a & b & c & d & e & 0 \\ 0 & 0 & 0 & a & b & c & d & e \\ A & B & C & D & E & 0 & 0 & 0 \\ 0 & A & B & C & D & E & 0 & 0 \\ 0 & 0 & A & B & C & D & E & 0 \\ 0 & 0 & 0 & A & B & C & D & E \end{vmatrix}$$

2811. Proposed by J. L. RILEY, Stephenville, Texas.

Given the cube roots of 60, 61, 63, and 64, to find the cube root of 62 by the method of differences.

2812. Proposed by C. N. SCHMALL, New York City.

If $F(x, y, z)$ be a homogeneous function of x, y, z, which becomes $\phi(u, v, w)$ by the elimination of x, y, z, by means of the equations $\partial F/\partial x = u$, $\partial F/\partial y = v$, $\partial F/\partial z = w$; show that

$$\frac{\partial F}{\partial u}\Big/x = \frac{\partial F}{\partial v}\Big/y = \frac{\partial F}{\partial w}\Big/z.$$

2813. Proposed by PAUL CAPRON, U. S. Naval Academy.

An ellipse having the major-axis $2a$ and the eccentricity ϵ, is revolved first about its major axis, forming a prolate spheroid, then about its minor axis forming an oblate spheroid. Show that the surfaces of these spheroids are, respectively,

$$2\pi a^2(1/\epsilon\sqrt{1-\epsilon^2}\sin^{-1}\epsilon + 1)$$

and

$$2\pi a^2\left[2 + 1/\epsilon(1-\epsilon^2)\log\left(\frac{1+\epsilon}{1-\epsilon}\right)\right].$$

SOLUTIONS OF PROBLEMS.

339 (Calculus) [June, 1913; May, 1919]. Proposed by T. H. GRONWALL, Washington, D. C.

To show that for any real value of x

$$\left|\frac{d^n}{dx^n}\left(\frac{\sin x}{x}\right)\right| \leqq \frac{1}{n+1}, \quad \text{and} \quad \left|\frac{d^n}{dx^n}\left(\frac{1-\cos x}{x}\right)\right| \leqq \frac{1}{n+1}.$$

I. SOLUTION BY OTTO DUNKEL, Washington University.

The function $y = \sin x/x$, $x \neq 0$; $y = 1$, $x = 0$, is single-valued and continuous for all values of x and it can be expressed as a power series

$$y = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots.$$

The power series shows that the function possesses all of its derivatives at $x = 0$, and it is easily seen from the power series or from the original form of the function that the derivatives exist for all other values of x. Hence, we may write the sequence of equations for finding the derivatives

$$xy = \sin x, \qquad y + xy' = \sin\left(\frac{\pi}{2} + x\right),$$

$$2y' + xy'' = \sin(\pi + x), \qquad \cdots, \qquad (n+1)y^{(n)} + xy^{(n+1)} = \sin\left(\frac{n+1}{2}\pi + x\right), \qquad \cdots.$$

These equations, taken in turn, show that $y, y', y'', \cdots$, approach zero as x becomes infinite. Thus $y^{(n)}$, for example, attains its maximum or minimum value for some finite value of x. For such a value of x we have $y^{(n+1)} = 0$ and

$$(n+1)y^{(n)} = \sin\left(\frac{n+1}{2}\pi + x\right)$$

and hence

$$|\, y^{(n)} \,| \leqq \frac{1}{n+1}.$$

The proof of the second inequality may be carried through in the same manner.

II. Solution by H. S. Uhler, Yale University.

Let $r \equiv (\sin x)/x$ and $s \equiv \sin x$, then

$$r = \frac{1}{x} \cdot s, \tag{0}$$

$$\frac{dr}{dx} + \frac{1}{x} \cdot r = \frac{1}{x} \cdot \frac{ds}{dx}, \tag{1}$$

$$\frac{d^2r}{dx^2} + \frac{2}{x} \cdot \frac{dr}{dx} = \frac{1}{x} \cdot \frac{d^2s}{dx^2}, \tag{2}$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\frac{d^{n-1}r}{dx^{n-1}} + \frac{n-1}{x} \cdot \frac{d^{n-2}r}{dx^{n-2}} = \frac{1}{x} \cdot \frac{d^{n-1}s}{dx^{n-1}}, \tag{$n-1$}$$

$$\frac{d^nr}{dx^n} + \frac{n}{x} \cdot \frac{d^{n-1}r}{dx^{n-1}} = \frac{1}{x} \cdot \frac{d^ns}{dx^n}. \tag{n}$$

In order to find an explicit formula for the general derivative of r with respect to x in terms of n, x, $\sin x$, and $\cos x$ we may proceed as follows:

Case 1. *n even.* Multiply equations (n), $(n-1)$, $(n-2)$, $(n-3)$, $\cdots$, (2), (1), (0) by $+1$

$$-\frac{n}{x}, \; +\frac{n}{x} \cdot \frac{n-1}{x}, \; -\frac{n}{x} \cdot \frac{n-1}{x} \cdot \frac{n-2}{x}, \; \cdots, \; +\frac{n}{x} \cdot \frac{n-1}{x} \cdots \frac{4}{x} \cdot \frac{3}{x}, \; -\frac{n}{x} \cdot \frac{n-1}{x} \cdots \frac{3}{x} \cdot \frac{2}{x},$$

$$+\frac{n}{x} \cdot \frac{n-1}{x} \cdots \frac{2}{x} \cdot \frac{1}{x},$$ respectively, and add the results. We then find

$$\frac{d^nr}{dx^n} = \pm\frac{1}{x} \cdot F_1(x) \cdot \sin x \pm \frac{n}{x^2} \cdot F_2(x) \cdot \cos x, \tag{A}$$

where

$$F_1(x) \equiv 1 - \frac{n(n-1)}{x^2} + \frac{n(n-1)(n-2)(n-3)}{x^4} - \cdots \mp \frac{n(n-1)\cdots 4\cdot 3}{x^{n-2}} \pm \frac{n(n-1)\cdots 2\cdot 1}{x^n},$$

$$F_2(x) \equiv 1 - \frac{(n-1)(n-2)}{x^2} + \frac{(n-1)(n-2)(n-3)(n-4)}{x^4} - \cdots \pm \frac{(n-1)(n-2)\cdots 5\cdot 4}{x^{n-4}} \mp \frac{(n-1)(n-2)\cdots 3\cdot 2}{x^{n-2}}.$$

The upper or lower signs are to be used throughout according as n is of the form $4m$ or $4m-2$, respectively; $m = 1, 2, 3, \cdots$. When $n = 4m$, the number of terms in $F_1(x)$ and $F_2(x)$ is $2m+1$ and $2m$, whereas when $n = 4m-2$ the number of terms in $F_1(x)$ and $F_2(x)$ is $2m$ and $2m-1$, respectively.

Case 2. *n odd.* Multiply equations (n), $(n-1)$, $\cdots$, (1), (0) by the same factors as were used above, observing that the signs of the last three factors will now be $-$, $+$, $-$; and add.

$$\frac{d^nr}{dx^n} = \mp\frac{1}{x} \cdot F_3(x) \cdot \cos x \pm \frac{n}{x^2} \cdot F_4(x) \cdot \sin x \tag{B}$$

where

$$F_3(x) \equiv 1 - \frac{n(n-1)}{x^2} + \frac{n(n-1)(n-2)(n-3)}{x^4} - \cdots \pm \frac{n(n-1)\cdots 5\cdot 4}{x^{n-3}} \mp \frac{n(n-1)\cdots 3\cdot 2}{x^{n-1}},$$

$$F_4(x) \equiv 1 - \frac{(n-1)(n-2)}{x^2} + \frac{(n-1)(n-2)(n-3)(n-4)}{x^4}$$

$$- \cdots \pm \frac{(n-1)(n-2)\cdots 4\cdot 3}{x^{n-3}} \mp \frac{(n-1)(n-2)\cdots 2\cdot 1}{x^{n-1}}.$$

The upper or lower signs are to be used throughout according as n is of the form $4m-1$ or $4m-3$, respectively. When $n = 4m-1$, or $n = 4m-3$ the number of terms in both $F_3(x)$ and $F_4(x)$ is $2m$ or $2m-1$, respectively. In all cases, the total number of terms in the right hand members of formulæ (A) and (B) is $n+1$.

The following properties of these formulæ will help to throw light on the general problem. Since $F_1(x)$, $F_2(x)$, $F_3(x)$, and $F_4(x)$ involve only even powers of x it is seen, at once, from the outstanding factors $\pm \frac{\sin x}{x}$, $\pm \frac{n\cos x}{x^2}$, and $\mp \frac{\cos x}{x}$, $\pm \frac{n \sin x}{x^2}$ that d^nr/dx^n, when numerically equal positive and negative values are substituted for x (n being always a positive integer), does not change sign for n even, and it does change sign for n odd. Hence the graph of $y = d^nr/dx^n$ is symmetrical with respect to the y-axis when n is even. On the other hand, the ordinates are equal in magnitude but opposite in sign for $x = +c$ and $x = -c$, when n is odd. When x is neither zero nor infinite it is evident that d^nr/dx^n is finite, continuous, and single-valued. By reducing all terms of the right-hand members of (A) and (B) to the common denominator x^{n+1} we see that d^nr/dx^n assumes the so-called indeterminate form $0/0$ when $x = 0$. Differentiating the re-written numerators of (A) and (B), and dividing each result by $d/dx(x^{n+1})$, *i.e.*, by $(n+1)x^n$, we find $\pm \frac{x^n \cos x}{(n+1)x^n}$ and $\pm \frac{x^n \sin x}{(n+1)x^n}$, respectively. Cancelling x^n and then substituting zero for x we obtain $\pm \frac{1}{n+1}$ and 0 for the limits of d^nr/dx^n corresponding, respectively, to even and odd values of n. Finally, since $|\sin x| \not> 1$, and $|\cos x| \not> 1$, (A) and (B) show that d^nr/dx^n vanishes as x becomes infinite, n remaining finite. It has thus been demonstrated that, for all values of x, d^nr/dx^n is a finite, continuous, and single-valued function of x.

The properties of d^nr/dx^n just proved show that the greatest numerical values of this derivative may be obtained from its algebraic maxima and minima, if it possesses any. Accordingly the next step in the argument is to establish the existence of maxima and minima.

A necessary condition that d^nr/dx^n shall have maxima and minima is $d^{n+1}r/dx^{n+1} = 0$. Hence it must be shown that the $(n+1)$th derivative can always vanish when neither n nor x is infinite. When n is even, $n+1$ is odd so that formula (B) applies. When n is odd, $n+1$ is even and formula (A) applies. It is not necessary to write out separately and in detail the equations for $d^{n+1}r/dx^{n+1} = 0$ since both assume the form

$$f(x)\cdot \sin x = xF(x)\cdot \cos x \qquad (C)$$

where $f(x)$ and $F(x)$ are polynomials in x^2, each having integral coefficients and a term free from x.

We are to show that the necessary condition (C) is an equation which always has real roots in addition to $x = 0$. Now $f(x) = 0$ and $F(x) = 0$ either have one or more real roots in common, or they have no common root. If such a root be admitted then a proof of the existence of a real root of condition (C) would be superfluous. Supposing that $f(x) = 0$ and $F(x) = 0$ have no common root we may demonstrate the existence of real roots of equation (C) in the following manner.

In the first place, assume that both $f(x) = 0$ and $F(x) = 0$ have real roots. Then take a positive value of x, R say, which is greater than the numerically largest root of $f(x) = 0$ and $F(x) = 0$. Also imagine the curves $y = f(x)\cdot \sin x$ and $y = xF(x)\cdot \cos x$ plotted to the same scale on the same diagram, and consider the possibility of the two loci having real intersections. For values of x between R and $+\infty$ the curves can cross the axis of x only when $\sin x$ and $\cos x$ vanish, for we have chosen R too large to permit either $f(x)$ or $F(x)$ to become zero. The curves $(x > R)$ intersect the x-axis at *alternate* points which have the constant interval $\pi/2$. In short, for $x > R$, the loci have the general nature of $y = \sin x$ and $y = \cos x$ except in so far as each trigonometric ratio is multiplied by a function of x that is finite, continuous, and single-valued ("damped" harmonic curves). It is clear, therefore, that for $x > R$ the loci intersect each other in an infinite number of discrete real points and hence that equation (C) has an infinite number of real roots.

The same argument applies if $f(x) = 0$ and $F(x) = 0$ have no real roots. In this case, however, we do not have to bother about R, as the possibility of one locus avoiding the other by crossing and recrossing the x-axis at the vanishing points of $f(x)$ and $F(x)$ no longer obtains.

For the sake of completeness the following comments seem appropriate. The question as to whether $f(x) = 0$ and $F(x) = 0$ have one or more common roots is not germane to the proof just presented. A similar remark applies to a discussion of the nature of the roots of these equations taken separately. Again, condition (C) cannot be fulfilled by the simultaneous vanishing of the members of the pairs $f(x)$, $\cos x$ and $F(x)$, $\sin x$. For, $\cos x$ and $\sin x$ vanish when, and only $(x \neq 0)$ when x is an integral multiple of $\pm (\pi/2)$ or $\pm \pi$, and it is a well-established fact,—upon which the proofs of the transcendentality of π depend (Lindemann, Hilbert),—that π cannot be a root of any "algebraic" equation, *a fortiori* for one having integral coefficients, *e.g.*, $f(x) = 0$ and $F(x) = 0$.

Having now established the existence of real roots of (C) we must next demonstrate the sufficiency of this condition, that is, we must show that $d^{n+2}r/dx^{n+2}$ cannot vanish when $d^{n+1}r/dx^{n+1} = 0$. The $(n + 2)$th equation of the set given at the very beginning is

$$\frac{d^{n+2}r}{dx^{n+2}} + \frac{n+2}{x} \cdot \frac{d^{n+1}r}{dx^{n+1}} = \frac{1}{x} \cdot \frac{d^{n+2}s}{dx^{n+2}}.$$

Therefore, when $d^{n+1}r/dx^{n+1} = 0$, we have also

$$\frac{d^{n+2}r}{dx^{n+2}} = \frac{1}{x} \cdot \frac{d^{n+2}s}{dx^{n+2}},$$

where $d^{n+2}s/dx^{n+2}$ has the values $-\sin x$, $+\cos x$, $+\sin x$, $-\cos x$ corresponding, respectively, to $n = 4m$, $4m - 1$, $4m - 2$, $4m - 3$. Now $(\cos x)/x$ cannot here vanish for a finite value of x because the necessary condition (C) is not fulfilled (as explained above) when $\cos x = 0$. Again $(\sin x)/x$ cannot vanish for a non-infinite value of x since, as before, condition (C) prevents $\sin x = 0$ for $x \neq 0$, and the ratio $(\sin x)/x$ approaches 1 for $x \doteq 0$. Consequently it has been shown that the necessary condition for the occurrence of maxima and minima of d^nr/dx^n is also sufficient.

We are now prepared to discuss the values of d^nr/dx^n at its maxima and minima. The $(n + 1)$th equation of the original list would be

$$\frac{d^{n+1}r}{dx^{n+1}} + \frac{n+1}{x} \cdot \frac{d^nr}{dx^n} = \frac{1}{x} \cdot \frac{d^{n+1}s}{dx^{n+1}}.$$

Hence, when $d^{n+1}r/dx^{n+1} = 0$ we have also

$$\frac{d^nr}{dx^n} = \frac{1}{n+1} \cdot \frac{d^{n+1}s}{dx^{n+1}},$$

where $d^{n+1}s/dx^{n+1}$ has the values $+\cos x$, $+\sin x$, $-\cos x$, $-\sin x$ associated, respectively, with $n = 4m$, $4m - 1$, $4m - 2$, $4m - 3$. Therefore, $|d^nr/dx^n|$ equals $1/(n + 1) \cdot |\cos x|$ or $1/(n + 1) \cdot |\sin x|$ when n is even or odd, respectively. Since $|\cos x| \leqq 1$ and $|\sin x| \leqq 1$ it follows that $|d^nr/dx^n|$ cannot exceed $1/(n + 1)$. As we have seen that (C) is satisfied by $x = 0$ but not by any integral multiple of either $\pm (\pi/2)$ or $\pm \pi$, the final results of the analysis may be written:

$$\left|\frac{d^nr}{dx^n}\right|_{x=0} = \frac{1}{n+1}, \text{ for } n \text{ even},$$

$$\left|\frac{d^nr}{dx^n}\right| < \frac{1}{n+1}, \text{ for } n \text{ odd, or for } n \text{ even and } x \neq 0.$$

REMARK. In the special case $n = 1$, I have followed the numerical values of

$$\frac{dr}{dx} = \frac{x \cos x - \sin x}{x^2}$$

from $x = 0$ through its greatest arithmetic maximum, which occurs when $x = \pm 2.081576\cdots$ (or $119° 15' 55.87''\cdots$). The corresponding value of $|dr/dx|$ is $0.436182\cdots$, which is about 12.8 per cent. less than $1/(n + 1) = 1/2$.

Attention must next be turned to

$$\frac{1-\cos x}{x} \equiv u.$$

Writing c in place of $1-\cos x$ the general equation is found at once to be

$$\frac{d^n u}{dx^n} + \frac{n}{x}\cdot\frac{d^{n-1}u}{dx^{n-1}} = \frac{1}{x}\cdot\frac{d^n c}{dx^n}.$$

Using precisely the same multipliers and method as before, we obtain

$$\frac{d^n u}{dx^n} = \mp\frac{1}{x}\cdot F_1(x)\cdot\cos x \pm \frac{n}{x^2}\cdot F_2(x)\cdot\sin x + \frac{\lfloor n}{x^{n+1}}, \qquad (A')$$

$$\frac{d^n u}{dx^n} = \mp\frac{1}{x}\cdot F_3(x)\cdot\sin x \mp \frac{n}{x^2}\cdot F_4(x)\cdot\cos x - \frac{\lfloor n}{x^{n+1}}. \qquad (B')$$

When n is even, (A'), and x is given pairs of values $+a$ and $-a$, $d^n u/dx^n$ assumes values that are equal in magnitude but opposite in sign. On the other hand, when n is odd this derivative alters neither in sign nor in magnitude when x is changed from $+a$ to $-a$. Hence, when n is odd $y = d^n u/dx^n$ is symmetrical with respect to the axis of ordinates.

Since all the separate terms of formulæ (A') and (B') involve either $\sin x$, or $\cos x$, or $\lfloor n$ in the numerator and some positive integral power of x in the denominator we see that $d^n u/dx^n$ is finite for all values of x between 0 and ∞. When x becomes infinite $d^n u/dx^n = 0$. When $x = 0$ the indeterminate form 0/0, when treated in the usual way, gives $\pm\dfrac{x^n \sin x}{(n+1)x^n}$ for n even, and $\mp\dfrac{x^n\cos x}{(n+1)x^n}$ for n odd. Therefore, for $x = 0$, the nth derivative of u is equal to 0 or $\mp\dfrac{1}{n+1}$ according as n is even or odd, respectively.

For sake of brevity and variety, the question of the existence of maxima and minima may be settled by making use of the elementary properties of plane curves. The fraction c/x is finite, continuous, single-valued, with an infinite number of maxima and minima. Whenever u has a maximum or a minimum du/dx vanishes. In other words, du/dx changes sign for every value of x that corresponds to a maximum or a minimum of u. Also the general formulæ (A') and (B') show that any derivative is finite, continuous, and single-valued. Therefore, du/dx must likewise have an infinite number of maxima and minima. We then apply the foregoing argument to du/dx and d^2u/dx^2, and so on indefinitely. Obviously the same line of reasoning could have been applied to r and all of its x-derivatives. The earlier proof was presented *in extenso* because of its rigor and probable instructive value.

Proceeding as in the case of r, we find that, when $d^{n+1}u/dx^{n+1} = 0$,

$$\frac{d^n u}{dx^n} = \frac{1}{n+1}\cdot\frac{d^{n+1}c}{dx^{n+1}}.$$

Accordingly, at a maximum or minimum,

$$\left|\frac{d^n u}{dx^n}\right| \text{ equals } \frac{1}{n+1}\cdot|\sin x| \quad \text{or} \quad \frac{1}{n+1}\cdot|\cos x|$$

according as n is even or odd. Finally we conclude that

$$\left|\frac{d^n u}{dx^n}\right|_{x=0} = \frac{1}{n+1}, \text{ for } n \text{ odd},$$

$$\left|\frac{d^n u}{dx^n}\right| < \frac{1}{n+1}, \text{ for } n \text{ even, or for } n \text{ odd and } x \neq 0.$$

2732 [1918, 444]. Proposed by PAUL CAPRON, U. S. Naval Academy.

A conical cup, filled with fluid, stands with the vertex upward on a smooth horizontal surface. The inner and outer surfaces of the cup are similar cones of revolution, having altitudes h and $h(1+x)$; the ratio of the specific weights of the material of the cone and the fluid is σ; the height of a barometer column of the fluid is h_0. Show that for equilibrium

$$\frac{h_0}{h}(1+x)^2 + \sigma x(1+x+x^2/3) < 2/3.$$

SOLUTION BY THE PROPOSER.

Let w be the specific weight of the fluid; then the atmospheric pressure is wh_0. Let a be the inner radius of the cone and P = the vertical component of the pressure between the fluid and the cone.

The weight of the fluid is $\frac{1}{3}\pi a^2 hw$, and

The weight of the cone is $\frac{1}{3}\pi a^2 h\sigma w[(1+x)^3 - 1]$.

The pressure between the horizontal surface and the cone is $\pi a^2 hw$.

The atmospheric pressure upon the cone has the vertical component $\pi a^2(1+x)^2 wh_0$.

From the equilibrium of the fluid,

$$P + \tfrac{1}{3}\pi a^2 hw = \pi a^2 hw.$$

From the equilibrium of the cone,

$$\tfrac{1}{3}\pi a^2 h\sigma w[(1+x)^3 - 1] + \pi a^2(1+x)^2 wh_0 \lessdot P = \tfrac{2}{3}\pi a^2 hw.$$

Hence, for equilibrium,

$$\frac{h_0}{h}(1+x)^2 + \sigma x(1 + x + x^2/3) \lessdot 2/3.$$

2733 [1918, 444]. Proposed by J. L. RILEY, Stephenville, Texas.

An ellipse of constant eccentricity passes through the focus of a parabola and has its foci on the curve. Find the envelopes of its axes.

SOLUTION BY WILLIAM HOOVER, Columbus, Ohio.

Let F be the focus, $(a, 0)$ of the parabola,

$$y^2 = 4ax; \tag{1}$$

F_1, F_2, the foci, (x_1, y_1), (x_2, y_2) of the ellipse in any one of its positions; $2A$, e, the major axis and constant eccentricity.

We have

$$FF_1 + FF_2 = 2A, \tag{2}$$

$FF_1 = x_1 + a$, and $FF_2 = x_2 + a$; and hence

$$2a + x_1 + x_2 = 2A. \tag{3}$$

Let the equation of the major axis of the ellipse be

$$y = mx + b. \tag{4}$$

Eliminate y from (4) and (1), and we have the quadratic giving the abscissas x_1, x_2 of F_1, F_2,

$$m^2x^2 - (4a - 2bm)x + b^2 = 0 \tag{5}$$

and so

$$x_1 + x_2 = (4a - 2bm)/m^2. \tag{6}$$

Similarly eliminating x,

$$my^2 - 4ay + 4ab = 0, \tag{7}$$

and then

$$y_1 + y_2 = 4a/m, \tag{8}$$

and (3) becomes

$$(2am^2 - 2bm + 4a)/m^2 = 2A. \tag{9}$$

Again,

$$(F_1F_2)^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 = 4A^2e^2. \tag{10}$$

From (5),

$$x_1x_2 = b^2/m^2, \tag{11}$$

and from (7),

$$y_1y_2 = 4ab/m. \tag{12}$$

From (6) and (11),

$$(x_1 - x_2)^2 = 16a(a - bm)/m^4, \tag{13}$$

and from (8) and (12).

$$(y_1 - y_2)^2 = 16a(a - bm)/m^2. \tag{14}$$

Substituting (13) and (14) in (10),

$$16a(1 + m^2)(a - bm)/m^4 = 4A^2e^2. \tag{15}$$

From (15) and (9) we obtain:

$$\frac{4a(1 + m^2)(a - bm)}{\{a(1 + m^2) + a - bm\}^2} = e^2. \tag{16}$$

An accommodating feature of this problem is the analytical form of the left member of (16). Let

$$a - bm = a(1 + m^2) \tan^2 \epsilon; \tag{17}$$

then (16) becomes

$$\frac{2 \tan \epsilon}{1 + \tan^2 \epsilon} = 2 \sin \epsilon \cos \epsilon = \sin 2\epsilon = e,$$

consistent with e not > 1 for the ellipse. Substituting b from (17) in (4), the equation of the major axis is the quadratic in m,

$$(x - a \tan^2 \epsilon)m^2 - ym + a(1 - \tan^2 \epsilon) = 0. \tag{18}$$

The condition for equal values of m in (18) is

$$y^2 = 4a(1 - \tan^2 \epsilon)(x - a \tan^2 \epsilon),$$

the envelope of the major axis.

The equation of the straight line through the center of the ellipse and perpendicular to (4) is

$$y - \frac{2a}{m} = -\frac{1}{m}\left(x - \frac{2a - bm}{m^2}\right);$$

or, substituting b from (17) and arranging,

$$ym^3 + (x - 2a - a \tan^2 \epsilon)m^2 - a \sec^2 \epsilon = 0. \tag{19}$$

Differentiating with respect to the variable parameter m, we find $m = -2(x - 2a - a \tan^2 \epsilon)/3y$ and (19) becomes

$$y^2 = \frac{4}{27a \sec^2 \epsilon}(x - a - a \sec^2 \epsilon)^3,$$

the envelope of the minor axis of the ellipse.

Also solved by the PROPOSER.

NOTES AND NEWS.

EDITED BY E. J. MOULTON, Northwestern University, Evanston, Ill.

Assistant Professor A. F. CARPENTER, of the University of Seattle, has been promoted to an associate professorship of mathematics.

At the University of California, Associate Professor C. A. NOBLE has been promoted to a full professorship of mathematics, and Assistant Professor B. M. WOODS to a full professorship of aërodynamics.

Mr. E. D. MEACHAM, of the University of Oklahoma, has been promoted to an assistant professorship of mathematics.

Assistant Professor H. H. CONWELL, of the University of Idaho, has been promoted to an associate professorship of mathematics.

Dr. R. A. ARMS, of Juniata College, has been appointed instructor in mathematics at the University of Pennsylvania.

Assistant Professor K. P. WILLIAMS, of Indiana University, after serving as captain with the Rainbow Division in France, has returned and been promoted to an associate professorship.

Associate Professor R. B. MCCLENON, of Grinnell College, has been promoted to a full professorship of mathematics.

Dr. J. V. DEPORTE has been appointed assistant professor of mathematics at the New York State College for Teachers, Albany.

Assistant Professor FLORENCE P. LEWIS, of Goucher College, has been promoted to an associate professorship of mathematics.

At Vassar College, Assistant Professor LOUISE D. CUMMINGS has been promoted to an associate professorship and Dr. MARY E. WELLS to an assistant professorship of mathematics.

Dr. J. W. CAMPBELL, of Wesley College, Winnipeg, has been appointed associate in mathematics and astronomy at the State University of Iowa.

New appointments at the University of Wisconsin for the present year, not previously announced, include: Messrs. C. D. EHRMAN, I. W. TRAYLER, J. E. DAVIS, H. L. OLSON, M. INGRAHAM, F. E. RECHART, and Misses E. M. F. FELTGES and B. ALBRIGHT to instructorships; and Messrs. T. M. DAHM, D. R. LAMONT, R. R. KNOEN, and D. J. STEWART to assistantships.

At the University of Minnesota, Mr. R. M. MATHEWS and Miss OLIVE ATWOOD have been appointed instructors in mathematics.

Assistant Professor T. E. MASON, of Purdue University, has been promoted to an associate professorship.

Dr. C. M. HEBBERT has been appointed instructor in mathematics at the University of Illinois.

Assistant Professor ARTHUR RAMSEY has been promoted to a professorship of mathematics in Grove City College.

Miss OTTILIA W. DUEKER has been appointed professor of mathematics and Dean of Women at Friends University, Wichita, Kansas.

Adjunct Professor J. J. LUCK, of the University of Virginia, has been promoted to an associate professorship of mathematics.

Professor F. J. HOLDER, of the University of Pittsburgh, has been appointed professor of mathematics and dean of the school of commerce at Mercer University, Macon, Ga.

Associate Professor R. P. STEPHENS, of the University of Georgia, has been promoted to a full professorship of mathematics.

Mr. J. J. TANZOLA has been appointed instructor in mathematics at Cooper Union.

Dr. C. C. BRAMBLE, of the U. S. Naval Academy, has been promoted to an assistant professorship of mathematics.

At the Armour Institute of Technology, assistant professor W. C. KRATHWOHL has been promoted to an associate professorship and Dr. W. L. MISER, of the University of Arkansas, has been appointed assistant professor of mathematics.

Dr. C. H. FORSYTH, of Dartmouth College, has been promoted to an assistant professorship of mathematics.

At the College of the City of New York, Mr. HARRY LANGMAN has been appointed instructor in statistics and Dr. H. F. MACNEISH instructor in mathematics.

Capt. L. L. BURCHNALL, scholar of Christ Church, Oxford, has been appointed lecturer in mathematics in the University of Durham.

Sir J. J. THOMSON, master of Trinity College, Cambridge, who recently resigned the Cavendish professorship of experimental physics, has been elected to the newly established professorship of physics. This professorship is without stipend, and will terminate with the tenure of office of the first professor unless the university determines otherwise.

Dr. J. PROUDMAN has been appointed professor of applied mathematics at the University of Liverpool.

Professor PAUL APPELL, dean of the faculty of sciences, Paris, has resigned from the office that he has held for sixteen years.

Dr. H. LEBESGUE, of the University of Paris, has been promoted to a professorship of the application of analysis to geometry. Professor LEBESGUE has been elected president of the Société mathématique de France for the year 1919.

At the University of Nancy T. GOT, professor in the Lycée Marseille, and M. LEAU, professor in the Lycée St. Louis, are each giving a course in mathematics.

The following have been appointed maîtres de conférences: J. CHAPELON and J. KAMPÉ DE FÉRIET at the University of Lille, M. JANET at the University of Grenoble, and P. HUMBERT and E. TURRIÈRE at the University of Montpellier. Dr. Humbert is to serve during the absence of Dr. A. Denjoy, who is at the University of Utrecht, and Dr. Turrière replaces Professor H. VILLAT, called to the University of Strassburg.

C. F. A. CURRIER, professor of history and political science at Mass. Institute of Technology since 1891 died Sept. 6, 1919, aged 62 years. "His natural bent was mathematics," and in early years he collaborated with Wentworth in preparing the "Teachers Solutions" for the series of mathematics text-books.

Dr. JAMES MACLAY, professor of mathematics at Columbia University since 1905, died on November 28, 1919, aged fifty-five years. He studied at the University of Berlin two years, received his doctorate from Columbia in 1899, was instructor in mathematics there 1895–1901, and adjoint professor 1901–1905.

Professor W. GROSS, of the University of Vienna, died October 29, 1918, at the age of thirty-two years.

The death is reported of Dr. J. KELLER of the technical school at Zurich.

Professor EMILE DUMONT, of the Institut Michot-Montgenost at Brussels, was killed in action in the late war.

Popular Astronomy reports that Dr. VIKTOR KNORRE, who was for many years an observer at the Berlin Observatory died on August 25, 1919, at the age of seventy-eight. He came of a family long connected with astronomy. His grandfather was an observer at Dorpat and his father was director of the observatory at Nikolajew in South Russia.

The deaths of the following professors are reported in *l'Enseignement mathématique* for October: ANTOINE GOB, of Athéné royal at Liège, at the age of fifty-one; E. BÖTTCHER, of the University of Leipzig, Aug. 5, 1919, at the age of seventy-two; O. DZIOBEK, 1919, and F. GRÄFE, December 2, 1918, both of the Charlottenburg Technical School and both at the age of sixty-three; E. NETTO of the University of Griessen, on May 2, 1919, at the age of seventy-two; R. STURM of the University of Breslau, on April 12, 1919, at the age of seventy-seven; K. T. REYE and J. WELLSTEIN, both formerly at the University of Strassburg, REYE, in June, 1919, at the age of eighty-one and WELLSTEIN on June 24, 1919, in his fiftieth year.

This MONTHLY announced the death of Professor H. G. ZEUTHEN (*1919*, 323 and 404) on the authority of *l'Enseignement mathématique* for June. He did not die till January 6, 1920.

Professor G. N. WATSON, of the University of Birmingham, has been elected fellow of the Royal Society of London.

Professor E. PASCAL, of the University of Naples, and Professor E. ALMANSI, of the University of Rome, have been elected national members of the Reale Accademia dei Lincei, of Rome. Professor G. FARRO, of the University of Paris, has been elected a corresponding member.

Professor E. GOURSAT, of the University of Paris, has been elected member of the Paris Academy of Sciences, in the section of mathematics, and Professor H. ANDOYER, of the Sorbonne, in the section of astronomy. Professor Goursat succeeds Professor E. PICARD, who has been chosen perpetual secretary of the academy, Professor E. COSSERAT, of the University of Toulouse, has been elected non-resident member of the academy.

Professor H. FEHR, of the University of Geneva, has been elected foreign correspondent of the Royal Society of Sciences of Liège.

Professor H. VON MANGOLDT was elected president of the Deutsche Mathematiker-Vereinigung for 1918–19. Professor FELIX KLEIN was honorary president for the same period, in honor of the fiftieth anniversary of his doctorate, celebrated on December 12, 1918; Professor KLEIN celebrated his seventieth birthday on April 25, 1919.

Professor J. M. TAYLOR, of Colgate University, recently completed fifty years as a teacher of mathematics in that institution. It was most fitting, therefore, that he should be selected as the one to be honored by the university in connection with its centennial celebration held in October, 1919. Only two honorary degrees were conferred on this occasion, namely Doctor of Science on Professor Taylor and Doctor of Laws on Elihu Root. Professor Taylor is the author of a series of mathematical texts for schools and colleges which have been widely used. Professor H. E. SLAUGHT reports that the calculus text of the series was being tried out in manuscript form when he was a sophomore at Colgate in 1881, and he wishes to testify that much of his inspiration and enthusiasm in mathematics may be traced to that course and others under Professor Taylor.

The following note is taken from *Nature* for December 18:

The appointment of Mr. G. H. HARDY, fellow and mathematical lecturer of Trinity College, Cambridge, to the Savilian professorship of geometry at Oxford reminds us that the present year marks the tercentenary of the foundation by Sir Henry Savile of the first university chairs of geometry and astronomy in Great Britain. Gresham in 1596 had inaugurated similar professorships in London, but the Gresham College never attained the importance it might have done, and London had to wait two centuries for her university. Both the famous Elizabethans, Gresham and Savile, performed valuable services for their Queen and country, and both were favourites at Court. Savile, who was born near Halifax, Yorkshire, in 1549 was, from 1585 until his death in 1622, warden of Merton College, Oxford, of which he had been made a fellow in 1570. He founded

the Savilian professorships in 1619, and the first holders of them were Briggs and Bainbridge, the former of whom had been the first Gresham professor of geometry. Briggs, who, like Savile, was born near Halifax, is best known for his notable works on logarithms and his intimacy with Napier, and the details of his life are generally familiar. Bainbridge did not rise to the same celebrity as his colleague, which may be partly accounted for by the fact that he was trained as a physician, and while Savilian professor of astronomy he was also Linacre reader in medicine.

A supplementary list (see *1919*, 420) of Degrees of Doctor of Philosophy in mathematics conferred in the academic year 1918–1919 is as follows:

University of California: CLYDE LYNNE EARLE WOLFE, "On the indeterminate cubic equation $x^3 + Dy^3 + D^2z^3 - 3Dxyz = 1$."

University of Illinois: JESSIE MARIE JACOBS, "The trilinear binary form as a cubic surface."

University of Indiana: GERTRUDE IONE MCCAIN, "Series of iterated linear fractional functions—character of the functions"; HAROLD EICHHOLTZ WOLFE, "A study of plane circle to circle transformations by means of tetracyclic coördinates."

University of Pennsylvania: WAYNE SENSENIG, "The invariant theory of involutions on conics."

At the fourth annual meeting of the Mathematical Association of America, Professor D. E. SMITH, of Columbia University, was elected president, and Professors HELEN A. MERRILL, of Wellesley College, and E. J. WILCZYNSKI, of University of Chicago, vice-presidents. The members of the Council, elected to serve for three years were Professors R. D. CARMICHAEL, E. R. HEDRICK, H. E. SLAUGHT, and J. W. YOUNG.

At the annual meeting of the American Mathematical Society in New York, December 30–31, 96 members were present and 38 papers read. The officers elected included the following: as vice-presidents, Professors C. N. HASKINS and R. G. D. RICHARDSON; as members of the council to serve until December, 1922, Professors T. H. HILDEBRANDT, E. KASNER, W. A. MANNING, and H. H. MITCHELL.

At the thirty-fourth annual meeting of the American Historical Association at Cleveland, Ohio, a "Conference on the History of Science" was held on December 31, 1919. The following papers were read: "History of Egyptian medicine" (illustrated) by Professor T. W. TODD, Medical School, Western Reserve University; "Peter of Abano, a medieval scientist" by Professor LYNN THORNDYKE, Western Reserve University; "The history of algebra" by Professor L. C. KARPINSKI, University of Michigan; "The problem of the history of science in the college curriculum" by Professor HENRY CREW, Northwestern University.

FOURTH ANNUAL MEETING OF THE MATHEMATICAL ASSOCIATION OF AMERICA.

The fourth annual meeting of the Association was held at Columbia University on Thursday and Friday, January 1 and 2, 1920, in conjunction with the annual meeting of the American Mathematical Society, which was held on Tuesday and Wednesday, December 30 and 31, 1919. There were 143 in attendance at the sessions, including the following 108 members of the Association:

C. R. ADAMS, Brown University.
JOSEPH ALLEN, College of the City of New York.
R. C. ARCHIBALD, Brown University.
C. S. ATCHISON, Washington and Jefferson College.

CLARA L. BACON, Goucher College.
RALPH BEATLEY, Horace Mann School for Boys.
A. A. BENNETT, University of Texas.
E. G. BILL, Dartmouth College.
C. L. BOUTON, Harvard University.
JOSEPH BOWDEN, Adelphi College.
H. S. BROWN, Hamilton College.
DANIEL BUCHANAN, Queen's University.
R. W. BURGESS, Brown University.

W. D. CAIRNS, Oberlin College.
MARY E. CASTER, Paterson, N. J.
J. A. CLARKE, W. Philadelphia High School for Boys.
J. B. COLEMAN, University of South Carolina.
G. M. CONWELL, N. Y. State College for Teachers.
J. L. COOLIDGE, Harvard University.
ELIZABETH B. COWLEY, Vassar College.
C. H. CURRIER, Brown University.

ELEANOR C. DOAK, Mount Holyoke College.
E. L. DODD, University of Texas.
FLETCHER DURELL, Lawrenceville School.

H. S. EVERETT, Bucknell University.

T. S. FISKE, Columbia University.
W. B. FITE, Columbia University.
J. A. FOBERG, Crane Junior College.
T. M. FOCKE, Case School of Applied Science.

A. S. GALE, University of Rochester.
W. V. N. GARRETSON, Rutgers College.
R. E. GILMAN, Brown University.
O. E. GLENN, University of Pennsylvania.
W. C. GRAUSTEIN, Harvard University.

W. M. HAMILTON, U. S. Nautical Almanac Office.
H. E. HAWKES, Columbia University.
OLIVE C. HAZLETT, Mount Holyoke College.
E. R. HEDRICK, University of Missouri.

A. A. HIMWICH, Physician, New York City.
ANNA M. HOWE, Dana Hall, Wellesley, Mass.
L. S. HULBURT, Johns Hopkins University.
W. A. HURWITZ, Cornell University.

DUNHAM JACKSON, University of Minnesota.
S. A. JOFFE, Asst. Actuary, Mutual Life Ins. Co.

EDWARD KASNER, Columbia University.

W. D. LAMBERT, Coast and Geodetic Survey.
MARCIA L. LATHAM, Hunter College.
D. D. LEIB, Connecticut College for Women.
FLORENCE P. LEWIS, Goucher College.
H. M. LUFKIN, Cornell University.

H. B. MITCHELL, Columbia University.
C. N. MOORE, University of Cincinnati.
FRANK MORLEY, Johns Hopkins University.
H. M. MORSE, Harvard University.
F. D. MURNAGHAN, Johns Hopkins University.

E. J. OGLESBY, New York University.
F. W. OWENS, Cornell University.
HELEN B. OWENS, Cornell University.

GEORGE PAASWELL, Civil Engineer, New York City.
ALEXANDER PELL, Bryn Mawr, Pa.
ANNA J. PELL, Bryn Mawr College.
S. S. PENN, New York, N. Y.
L. R. PERKINS, Middlebury College.

PATRICK RAFFERTY, Holy Cross College.
W. W. RANKIN, Jr., University of South Carolina.
W. R. RANSOM, Tufts College.
H. W. REDDICK, Cooper Union.
L. J. REED, Johns Hopkins University.
C. N. REYNOLDS, Jr., Wesleyan University.
J. B. REYNOLDS, Lehigh University.
R. G. D. RICHARDSON, Brown University.
L. H. RICE, Tufts College.

PAUL SAUREL, College of the City of New York.
RALEIGH SCHORLING, Lincoln School.
W. H. SHERK, University of Buffalo.
L. P. SICELOFF, Columbia University.
W. G. SIMON, Western Reserve University.
LAO G. SIMONS, Hunter College.